CONTROL SYSTEMS

Department of Electronics & communication Engineering

Unit-1

1. Basic Concept of Control System

Control Engineering is concerned with techniques that are used to solve the following six problems in the most efficient manner possible.

(a)The identification problem :to measure the variables and convert data for analysis.

(b)The representation problem:to describe a system by an analytical form or mathematical model

(c)The solution problem:to determine the above system model response.

(d)The stability problem:general qualitative analysis of the system

(e)The design problem: modification of an existing system or develop a new one

(f)The optimization problem: from a variety of design to choose the best.

The two basic approaches to solve these six problems are conventional and modern approach. The electrical oriented conventional approach is based on complex function theory. The modern approach has mechanical orientation and based on the state variable theory.

Therefore, control engineering is not limited to any engineering discipline but is equally applicable to aeronautical, chemical, mechanical, environmental, civil and electrical engineering. For example, a control system often includes electrical, mechanical and chemical components.
Furthermore, as the understanding of the dynamics of business, social and political systems increases; the ability to control these systems will also increase.

1.1. Basic terminologies in control system

System: A combination or arrangement of a number of different physical components to form a whole unit such that that combining unit performs to achieve a certain goal.

Control: The action to command, direct or regulate a system.

Plant or process: The part or component of a system that is required to be controlled.

Input: It is the signal or excitation supplied to a control system.

Output: It is the actual response obtained from the control system.

Controller: The part or component of a system that controls the plant.

Disturbances: The signal that has adverse effect on the performance of a control system.

Control system: A system that can command, direct or regulate itself or another system to achieve a certain goal.

Automation: The control of a process by automatic means

Control System: An interconnection of components forming a system configuration that will provide a desired response.

Actuator: It is the device that causes the process to provide the output. It is the device that provides the motive power to the process.

Design: The process of conceiving or inventing the forms, parts, and details of system to achieve a specified purpose.
Simulation: A model of a system that is used to investigate the behavior of a system by utilizing actual input signals**.**

Optimization: The adjustment of the parameters to achieve the most favorable or advantageous design.

Feedback Signal: A measure of the output of the system used for feedback to control the system.

Negative feedback: The output signal is feedback so that it subtracts from the input signal.

Block diagrams: Unidirectional, operational blocks that represent the transfer functions of the elements of the system.

Signal Flow Graph (SFG): A diagram that consists of nodes connected by several directed branches and that is a graphical representation of a set of linear relations.

Specifications: Statements that explicitly state what the device or product is to be and to do. It is also defined as a set of prescribed performance criteria.

Open-loop control system: A system that utilizes a device to control the process without using feedback. Thus the output has no effect upon the signal to the process.

Closed-loop feedback control system: A system that uses a measurement of the output and compares it with the desired output.

Regulator: The control system where the desired values of the controlled outputs are more or less fixed and the main problem is to reject disturbance effects.

Servo system: The control system where the outputs are mechanical quantities like acceleration, velocity or position.

Stability: It is a notion that describes whether the system will be able to follow the input command. In a non-rigorous sense, a system is said to be unstable if its output is out of control or increases without bound.

Multivariable Control System: A system with more than one input variable or more than one output variable.

Trade-off: The result of making a judgment about how much compromise must be made between conflicting criteria.

1.2. Classification

1.2.1. Natural control system and Man-made control system:

Natural control system: It is a control system that is created by nature, i.e. solar system, digestive system of any animal, etc.

Man-made control system: It is a control system that is created by humans, i.e. automobile, power plants etc.

1.2.2. Automatic control system and Combinational control system:

Automatic control system: It is a control system that is made by using basic theories from mathematics and engineering. This system mainly has sensors, actuators and responders.

Combinational control system: It is a control system that is a combination of natural and man-made control systems, i.e. driving a car etc.

1.2.3. Time-variant control system and Time-invariant control system:

Time-variant control system: It is a control system where any one or more parameters of the control system vary with time i.e. driving a vehicle.

Time-invariant control system: It is a control system where none of its parameters vary with time i.e. control system made up of inductors, capacitors and resistors only.

1.2.4. Linear control system and Non-linear control system:

Linear control system: It is a control system that satisfies properties of homogeneity and additive.

Homogeneous property: $f\ x \quad y \quad f\ x \quad f\ y$

Additive property: $f \quad xf\ x$

Non-linear control system: It is a control system that does not satisfy properties of homogeneity and additive, i.e. $f\ x\ x^3$

1.2.1.10 Continuous-Time control system and Discrete-Time control system:

Continuous-Time control system: It is a control system where performances of all of its parameters are function of time, i.e. armature type speed control of motor.

Discrete -Time control system: It is a control system where performances of all of its parameters are function of discrete time i.e. microprocessor type speed control of motor.

1.2.1.11. Deterministic control system and Stochastic control system:

Deterministic control system: It is a control system where its output is predictable or repetitive for certain input signal or disturbance signal.

Stochastic control system: It is a control system where its output is unpredictable or non-repetitive for certain input signal or disturbance signal.

1.2.1.12. Lumped-parameter control system and Distributed-parameter control system:

Lumped-parameter control system: It is a control system where its mathematical model is represented by ordinary differential equations.

Distributed-parameter control system: It is a control system where its mathematical model is represented by an electrical network that is a combination of resistors, inductors and capacitors.

1.2.2. Single-input-single-output (SISO) control system and Multi-input-multi-output (MIMO) control system:

SISO control system: It is a control system that has only one input and one output.

MIMO control system: It is a control system that has only more than one input and more than one output.

1.2.9. Open-loop control system and Closed-loop control system:

Open-loop control system: It is a control system where its control action only depends on input signal and does not depend on its output response.

Closed-loop control system: It is a control system where its control action depends on both of its input signal and output response.

1.3. Open-loop control system and Closed-loop control system

1.3.1. Open-loop control system:

It is a control system where its control action only depends on input signal and does not depend on its output response as shown in Fig.1.1.

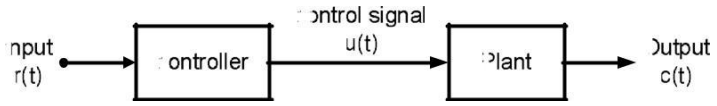


Fig.1.1. An open-loop system

Examples: traffic signal, washing machine, bread toaster, etc.

Advantages:

Simple design and easy to construct Economical
Easy for maintenance
Highly stable operation

Dis-advantages:

Not accurate and reliable when input or system parameters are variable in nature
Recalibration of the parameters are required time to time

1.3.2. Closed-loop control system:

It is a control system where its control action depends on both of its input signal and output response as shown in Fig.1.2.

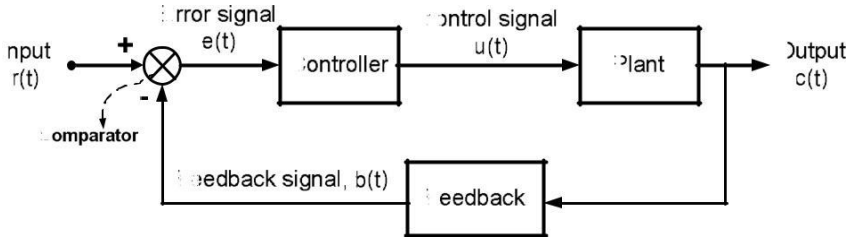


Fig.1.2. A closed-loop system

Examples: automatic electric iron, missile launcher, speed control of DC motor, etc.

Advantages:

More accurate operation than that of open-loop control system
Can operate efficiently when input or system parameters are variable in nature
Less nonlinearity effect of these systems on output response High bandwidth of operation
There is facility of automation
Time to time recalibration of the parameters are not required

Dis-advantages:

Complex design and difficult to construct

Expensive than that of open-loop control
system Complicate for maintenance
Less stable operation than that of open-loop control system

1.3.3. Comparison between Open-loop and Closed-loop control systems:

It is a control system where its control action depends on both of its input signal and output response.

Sl. No.	Open-loop control systems	Closed-loop control systems
1	No feedback is given to the control system	A feedback is given to the control system
2	Cannot be intelligent	Intelligent controlling action
3	There is no possibility of undesirable system oscillation(hunting)	Closed loop control introduces the possibility of undesirable system oscillation(hunting)
4	The output will not very for a constant input, provided the system parameters remain unaltered	In the system the output may vary for a constant input, depending upon the feedback
5	System output variation due to variation in parameters of the system is greater and the output very in an uncontrolled way	System output variation due to variation in parameters of the system is less.
6	Error detection is not present	Error detection is present
7	Small bandwidth	Large bandwidth
8	More stable	Less stable or prone to instability
9	Affected by non-linearities	Not affected by non-linearities
10	Very sensitive in nature	Less sensitive to disturbances
11	Simple design	Complex design
12	Cheap	Costly

1.4. Servomechanism

It is the feedback unit used in a control system. In this system, the control variable is a mechanical signal such as position, velocity or acceleration. Here, the output signal is directly fed to the comparator as the feedback signal, b(t) of the closed-loop control system. This type of system is used where both the command and output signals are mechanical in nature. A position control system as shown in Fig.1.3 is a simple example of this type mechanism. The block diagram of the servomechanism of an automatic steering system is shown in Fig.1.4.

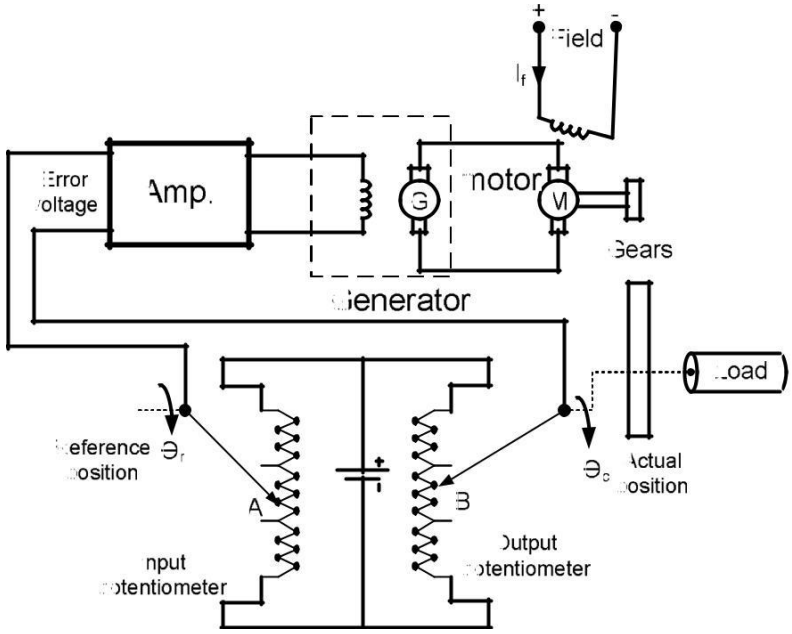


Fig.1.3. Schematic diagram of a servomechanism

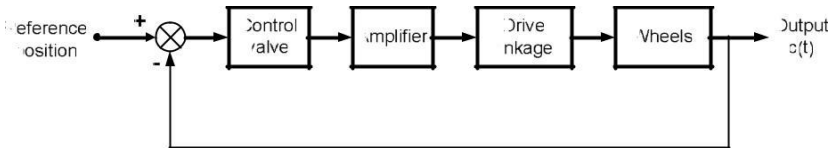


Fig.1.4. Block diagram of a servomechanism

Examples:

Missile launcher
Machine tool position control
Power steering for an automobile
Roll stabilization in ships, etc.

1.1.10 Regulators

It is also a feedback unit used in a control system like servomechanism. But, the output is kept constant at its desired value. The schematic diagram of a regulating

system is shown in Fig.1.1.10 Its corresponding simplified block diagram model is shown in Fig.1.1.11.

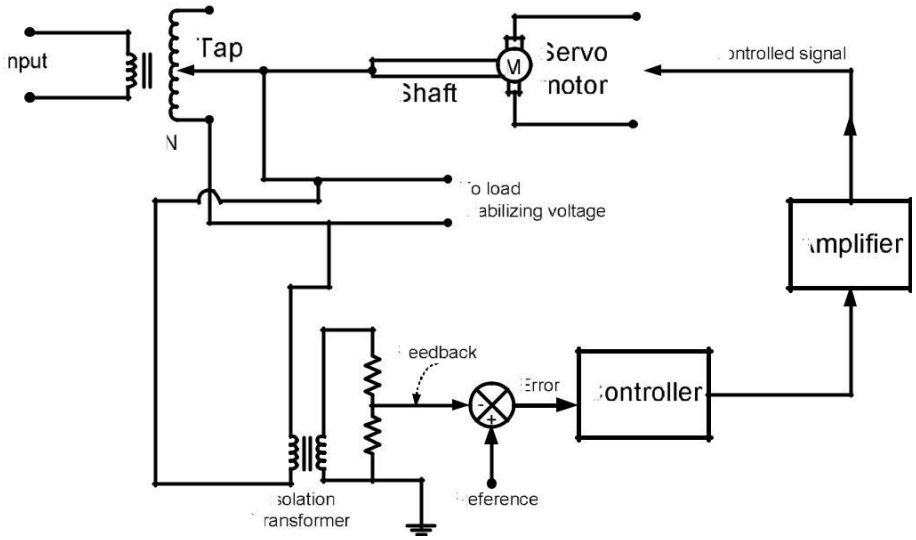


Fig.1.1.10 Schematic diagram of a regulating system

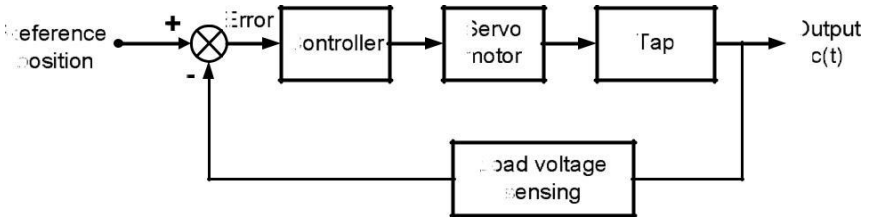


Fig.1.1.11. Block diagram of a regulating system

Examples:

Temperature regulator
Speed governor
Frequency regulators, etc.

Description of physical system

1.1.11. Components of a mechanical system: Mechanical systems are of two types, i.e. (i) translational mechanical system and (ii) rotational mechanical system.

1.1.11.1. Translational mechanical system

There are three basic elements in a translational mechanical system, i.e. (a) mass, (b) spring and (c) damper.

(a) **Mass:** A mass is denoted by M. If a force *f* is applied on it and it displays distance *x*, then $f = M \frac{d^2x}{dt^2}$ as shown in Fig.1.1.11.

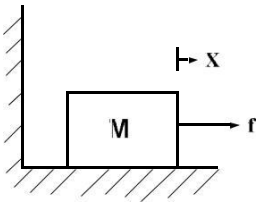


Fig.1.1.11. Force applied on a mass with displacement in one direction

If a force *f* is applied on a massM and it displays distance $x_1$in the direction of *f* and distance x_2 in the opposite direction, then $f = M\frac{d^2x_1}{dt^2} - \frac{d^2x_2}{dt^2}$ as shown in Fig.1.1.12.

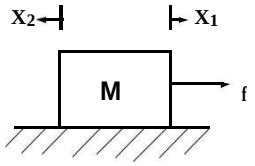


Fig.1.1.12. Force applied on a mass with displacement two directions

(b) **Spring:** A spring is denoted by K. If a force *f* is applied on it and it displays distance *x*, then $f = Kx$ as shown in Fig.1.2.

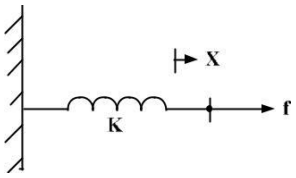


Fig.1.2. Force applied on a spring with displacement in one direction

If a force *f* is applied on a springK and it displays distance $x_1$in the direction of *f* and distance x_2 in the opposite direction, then $f = K(x_1 - x_2)$ as shown in Fig.4.4.

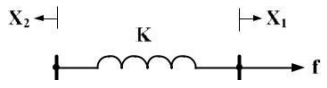


Fig.4.4. Force applied on a spring with displacement in two directions

(c) **Damper:** A damper is denoted by D. If a force *f* is applied on it and it displays distance *x*, then $f \; D \frac{dx}{dt}$ as shown in Fig.4.1.10

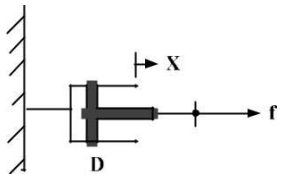


Fig.4.1.10 Force applied on a damper with displacement in one direction

If a force *f* is applied on a damperD and it displays distance x_1 in the direction of *f* and distance x_2 in the opposite direction, then $f \quad D\frac{dx_1}{dt} \quad \frac{dx_2}{dt}$ as shown in Fig.4.1.11.

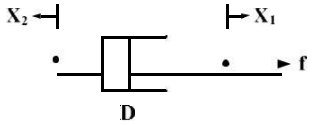


Fig.4.1.11. Force applied on a damper with displacement in two directions

1.1.11.2. Rotational mechanical system

There are three basic elements in a Rotational mechanical system, i.e. (a) inertia, (b) spring and (c) damper.

(a) **Inertia:** A body with aninertia is denoted by *J*. If a torque*T* is applied on it and it displays distanceΘ, then $T \quad J\frac{d^2}{dt^2}$. If a torque*T* is applied on a body with inertia *J* and it displays distance Θ_1 in the direction of *T* and distance Θ_2 in the opposite direction, then $T \; J\frac{d^2}{dt^2}_1 \quad \frac{d^2}{dt^2}_2$.

(b) **Spring:** A spring is denoted by K. If a torque*T* is applied on it and it displays distanceΘ, then $T \; K$. If a torque*T* is applied on a body with inertia *J* and it displays distance Θ_1 in the direction of *T* and distance Θ_2 in the opposite direction, then $T \; K_{1\,2}$.

(c) **Damper:** A damper is denoted by D. If a torque*T* is applied on it and it displays distanceΘ, then $T \; D\frac{d}{dt}$. If a torque*T* is applied on a body with inertia *J* and it

displays distance θ_1 in the direction of T and distance θ_2 in the opposite direction, then $T = D\frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}$.

1.1.12. Components of an electrical system: There are three basic elements in an electrical system, i.e. (a) resistor (R), (b) inductor(L) and (c) capacitor (C). Electrical systems are of two types, i.e. (i) voltage source electrical system and (ii) current source electrical system.

1.1.12.1. Voltage source electrical system: If i is the current through a resistor(Fig.4.7) and v is the voltage drop in it, then $v = Ri$.

If i is the current through an inductor (Fig.4.7) and v is the voltage developed in it, then $v = L\frac{di}{dt}$.

If i is the current through a capacitor(Fig.4.7) and v is the voltage developed in it, then $v = C\frac{1}{}\int idt$.

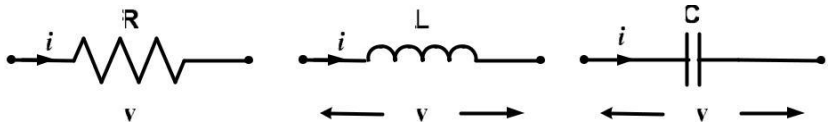


Fig.4.1.12. Current and voltage shown in resistor, inductor and capacitor

1.1.12.2. Current source electrical system:

If i is the current through a resistor and v is the voltage drop in it, then $i = \frac{v}{R}$.

If i is the current through an inductor and v is the voltage developed in it, then $i = \frac{1}{L}\int vdt$.

If i is the current through a capacitor and v is the voltage developed in it, then $i = C\frac{dv}{}$.

1.1.12.3. Work out problems:

Q.1.1.11. Find system transfer function betweenvoltage drop across the capacitanceand input voltage in the followingRC circuit as shown in Fig.4.2.

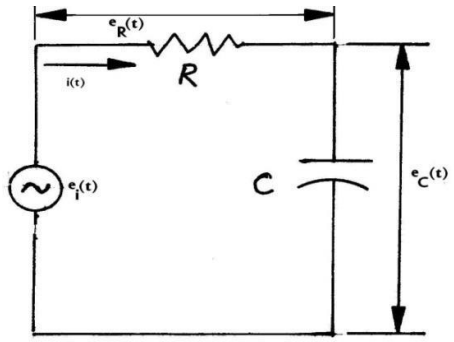


Fig.4.2.

Solution

Voltage across resistance, $e_R(t)$ $i(t)R$

Voltage across capacitance, $e_C(t)$ $\frac{1}{C}\int i(t)dt$

Total voltage drop, e_i e_R e_C $i(t)R$ $\frac{1}{C}\int i(t)dt$

Laplace transform of above equation, $E_i(s)$ $I(s)R$ $\frac{1}{Cs}$

System transfer function betweenvoltage drop across the capacitanceand input voltage, $\frac{E_C(s)}{E_i(s)}$ $\frac{1}{RCs+1}$ $\frac{1}{s+1}$

where, RC is the time-constant

Q.1.1.12. Find system transfer function betweenfunction between the inductance currentto the source currentin the followingRL circuit as shown in Fig.4.9.

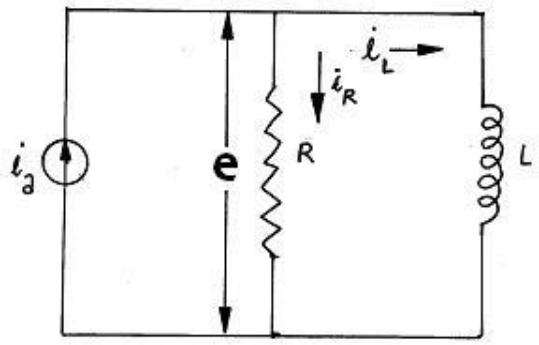


Fig.4.9.

Voltage across the Resistance, $e(t)$ $i_R R$ i_R $\frac{e(t)}{R}$

Voltage across the Inductance, $e(t)$ $L\frac{di_L}{dt}$ i_L $\frac{1}{L}\int e(t)dt$

Total current, i_a i_R i_L $\frac{e(t)}{R}$ $\frac{1}{L}\int e(t)dt$

Laplace transform of the current source,

$I_a(s)$ $E(s)\frac{1}{R}$ $\frac{1}{Ls}$ and $I_L(s)$ $\frac{E}{Ls}$

Transfer function between the inductance current to the source current,

$\frac{I_L(s)}{I_a(s)}$ $\frac{1}{\frac{L}{R}s+1}$ $\frac{1}{s+1}$

where $\frac{L}{R}$ is the time-constant

Q.1.2. Find system transfer function betweenfunction between the capacitance voltageto the source voltage in the followingRLC circuit as shown in Fig.1.60.

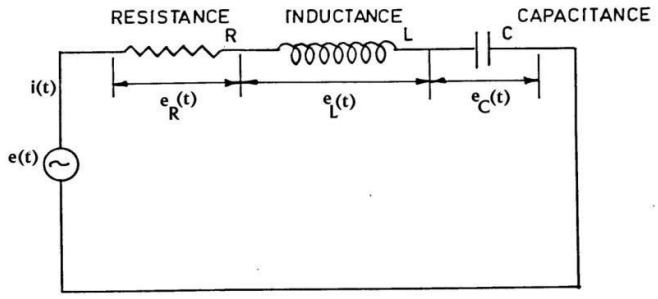


Fig.1.60.

Voltage across the Resistance, $e_R(t) \; iR$

Voltage across the Inductance, $e_L(t)$ $L\frac{di}{dt}$

Voltage across thecapacitance, $e_C(t) \; \frac{1}{C}\int idt$

Total voltage, $e\,t \;\; iR \;\; L\frac{di}{dt} \;\; \frac{1}{C}\int idt$

Laplace transform of the voltage source, $E(s) \; I(s)\,R \;\; Ls \;\; \frac{1}{Cs}$

Transfer function between capacitance voltage and source voltage

$$\frac{E_C(s)}{E(s)} \quad \frac{1}{Cs\,R \;\; Ls\frac{1}{Cs}} \quad \frac{\omega_n^2}{s^2 \;\; 2\,s\,\omega_n \;\; \omega_n^2}$$

where $\omega_n \;\; \frac{1}{\sqrt{LC}}$ and $\frac{R}{2\sqrt{\frac{L}{C}}}$

Q.. Find the transfer function of the following Spring-mass-damperas shown in Fig.1.61.

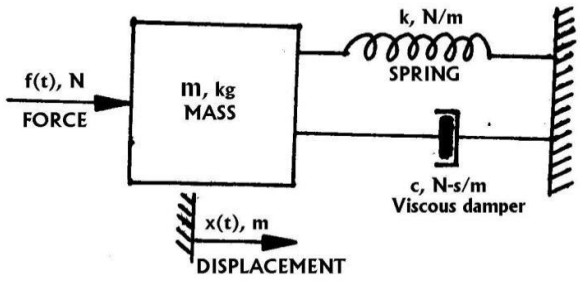


Fig.1.61.

Solution

$$\frac{X(s)}{F(s)} \quad \frac{1}{ms^2 \quad cs \quad k} \quad \frac{1}{m s^2 \quad 2 \ _n s \quad _n^2}$$

1.2. Analogous system: Fig.1.62 shows a translational mechanical system, a rotational control system and a voltage-source electrical system.

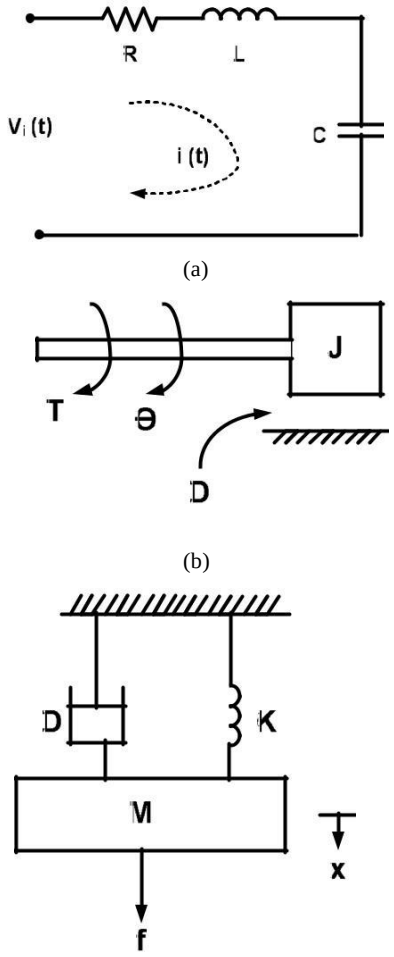


Fig.1.62. (a) a voltage-source electrical system,(b) a translational mechanical system and (c) a rotational control system

From Fig1.62 (a), (b) and (c), we have

$$\begin{aligned} L\frac{d^2q}{dt^2} R \quad \frac{dq}{dt} \quad \frac{1}{C} q \quad v\,t \\ J\frac{d^2}{dt^2} D \quad \frac{d}{dt} \quad K \quad T \\ M\frac{d^2x}{dt^2} D \quad \frac{dx}{dt} \quad Kx \quad f \end{aligned}$$

Equation Chapter 8 Section 0(1.6)

Where,

$$q \quad idt \tag{1.7}$$

The solutions for all the above three equations given by eq (1.7) are same. Therefore, the above shown three figures are analogous to each other. There are two important types of analogous systems, i.e. force-voltage (f-v) analogy and force-current analogy. From eq (1.7), f-v analogy can be drawn as follows.

Translational	Rotational	Electrical
Force (f)	Torque (T)	Voltage (v)
Mass (M)	Inertia (J)	Inductance (L)
Damper (D)	Damper (D)	Resistance (R)
Spring (K)	Spring (K)	Elastance (1/C)
Displacement (x)	Displacement (Θ)	Charge (q)
Velocity (u) = x	Velocity (u) =	

Similarly, f-i analogy that can be obtainedfrom eq (1.6), can be drawn as follows.

Translational	Rotational	Electrical
Force (f)	Torque (T)	Current (i)
Mass (M)	Inertia (J)	Capacitance (C)
Damper (D)	Damper (D)	Conductance (1/R)
Spring (K)	Spring (K)	Reciprocal of Inductance (1/L)
Displacement (x)	Displacement (Θ)	Flux linkage (ψ)
Velocity (u) = x	Velocity (u) =	Voltage (v) =

1.9. Mathematical model of armature controlled DC motor: The armature control type speed control system of a DC motor is shown in Fig.4.1.11. The following components are used in this system.

R_a=resistance of armature

L_a=inductance of armature winding

i_a=armature current

I_f=field current

E_a=applied armature voltage

E_b=back emf

T_m=torque developed by motor

Θ=angular displacement of motor shaft

J=equivalent moment of inertia and load referred to motor shaft

f=equivalent viscous friction coefficient of motor and load referred to motor shaft

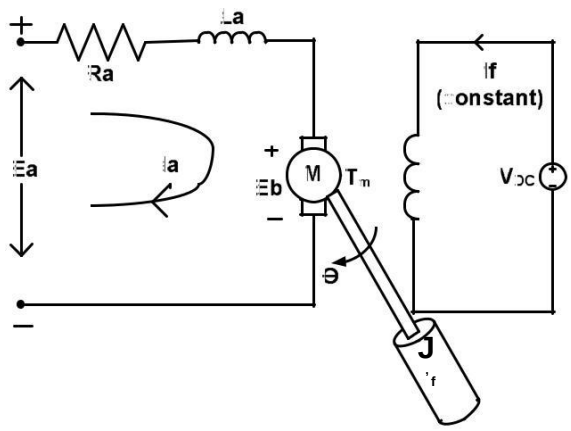


Fig.4.1.11. Schematic diagram of armature control type speed control system of a DC motor

The air-gap flux is proportional of the field current i.e.

$$K_f I_f \tag{1.8}$$

The torque T_m developed by the motor is proportional to the product of armature current and air gap flux i.e.

$$T_m = k_1 K_f I_f i_a \tag{4.4}$$

In armature-controlled D.C. motor,the field current is kept constant,so that eq(4.4) can be written as follows.

$$T_m = K_t i_a \tag{4.5}$$

The motor back emf being proportional to speed is given as follows.

$$E_b = K_b \frac{d}{dt} \tag{4.6}$$

The differential equation of the armature circuit is

$$L_a \frac{di_a}{dt} + R_a i_a = E_a - E_b \tag{4.7}$$

The torque equation is

$$J \frac{d^2}{dt^2} + f \frac{d}{dt} = T_m = K_t I_a \tag{4.8}$$

Taking the Laplace transforms of equations (4.6), (4.7) and (4.8), assuming zero initial conditions, we get

$$E_b(s) = sK_b(s) \tag{4.9}$$

$$sL_a + R_a I_a(s) = E_a(s) - E_b(s) \tag{1.60}$$

$$(s^2 J + sf)(s) = T_m(s) = K_t I_a \tag{1.61}$$

From eq(4.9) to (1.61) the transfer function of the system is obtained as,

$$G(s) = \frac{\Theta(s)}{E_a(s)} = \frac{K_t}{s\left[(R_a + sL_a)(sJ + f) + K_t K_b\right]} \tag{1.62}$$

Eq(1.62) can be rewritten as

$$G(s) = \frac{\Theta(s)}{E_a(s)} = \frac{\dfrac{K_t}{(R_a + sL_a)(sJ + f)}}{1 + \dfrac{K_t K_b}{(R_a + sL_a)(sJ + f)}} \cdot \frac{1}{s} \tag{1.63}$$

The block diagram that is constructed from eq (1.63) is shown in Fig.4.1.12.

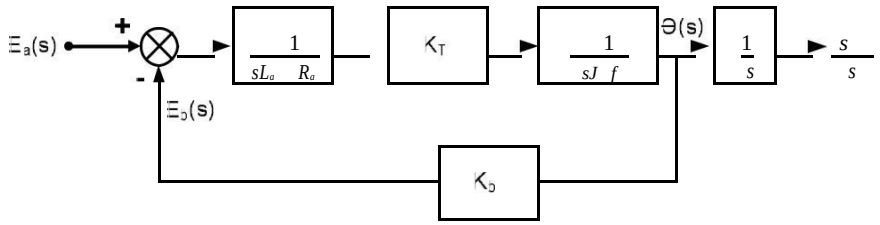


Fig.4.1.12. Block diagram of armature control type speed control system of a DC motor

The armature circuit inductance L_a is usually negligible. Therefore, eq(1.63) can be simplifiedas follows.

$$\frac{\Theta(s)}{E_a(s)} s^2 = \frac{K_t}{R_a} \quad J s + f + \frac{K_t K_b}{R_a} \tag{1.64}$$

The term $f + \frac{K_t K_b}{R_a}$ indicates that the back emf of the motor effectively increases the viscous friction of the system. Let,

$$f + f = \frac{K_t K_b}{R_a} \tag{1.65}$$

Where f be the effective viscous friction coefficient. The transfer function given by eq(1.65) may be written in the following form.

$$\frac{s}{E_a} = \frac{K_m}{s(s+1)} \tag{1.66}$$

Here $K_m = \frac{K_t}{R_a f}$ = motor gain constant, and $\frac{J}{f}$ = motor time constant.Therefore, the motor torque and back emf constant K_t, K_b are interrelated.

1.3.6Mathematical model of field controlled DC motor: The field control type speed control system of a DC motor is shown in Fig.4.2. The following components are used in this system.

R_f=Field winding resistance

L_f=inductance of field winding

I_f=field current

e_f=field control voltage

T_m=torque developed by motor

Θ=angular displacement of motor shaft

J=equivalent moment of inertia and load referred to motor shaft

f=equivalent viscous friction coefficient of motor and load referred to motor shaft

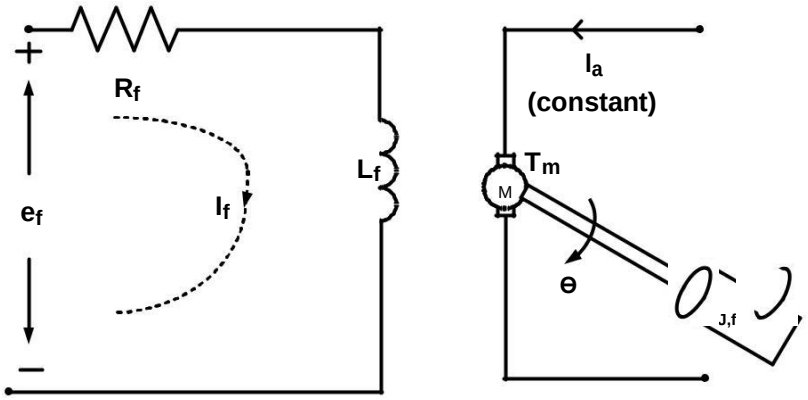


Fig.4.2. Block diagram of field control type speed control system of a DC motor

In field control motor the armature current is fed from a constant current source.The air-gap flux Φ is proportional of the field current i.e.

$$K_f I_f \tag{1.67}$$

The torque Tm developed by the motor is proportional to the product of armature current and air gap flux i.e.

$$T_m = k_1 K_f I_f I_a \quad K_t I_f \tag{1.68}$$

The equation for the field circuit is

$$L_f \frac{dI_f}{dt} R_f I_f \quad E_f \tag{1.69}$$

The torque equation is

$$J \frac{d^2}{dt^2} \quad f \quad \frac{d}{dt} \quad T \quad K \quad I \quad {}_{m} \quad {}_{t} \quad {}_{f} \tag{1.70}$$

Taking the Laplace transforms of equations (1.69) and (1.70) assuming zero initial conditions, we get the following equations

$$L_f s + R_f I_f(s) = E_f(s) \tag{1.71}$$

and

$$Js^2 + fs\,\theta(s) = T_m(s) = K_t I_f(s) \tag{1.72}$$

From eq(1.71) and (1.72) the transfer function of the system is obtained as

$$G(s) = \frac{\theta(s)}{E_f(s)} = \frac{K_t}{s(R_f + sL_f)(Js + f)} \tag{1.73}$$

The transfer function given by eq(1.73) may be written in the following form.

$$\frac{\theta(s)}{E_a(s)} = \frac{K_t}{sL_f(s + R_f/L_f)(Js + f)} = \frac{K_m}{s(s + 1/\tau_f)(1 + s\tau_m)} \tag{1.74}$$

Here $K_m = \frac{K_t}{R_f f}$ = motor gain constant, and $\frac{L_f}{R_f}$ = time constant of field circuit and $\frac{J}{f}$ = mechanical time constant.For small size motors field control is advantageous.The block diagram that is constructed from eq (1.74) is shown in Fig.4.9.

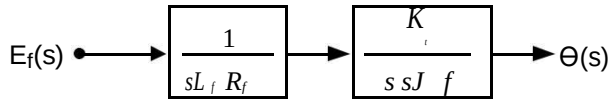


Fig.4.9. Block diagram of field control type speed control system of a DC motor

Block Diagram Algebra

1.101. Basic Definition in Block Diagram model:

Block diagram: It is the pictorial representation of the cause-and-response relationship between input and output of a physical system.

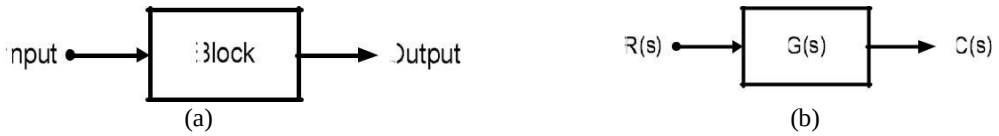


Fig.1.101. (a) A block diagram representation of a system and (b) A block diagram representation with gain of a system

Output: The value of input multiplied by the gain of the system.

$$C\,s \quad G\,s\,R\,s \tag{1.101}$$

Summing point: It is the component of a block diagram model at which two or more signals can be added or subtracted. In Fig.15, inputs R(s) and B(s) have been given to a summing point and its output signal is E(s). Here,

$$E\,s \quad R\,s \quad B\,s \tag{1.102}$$

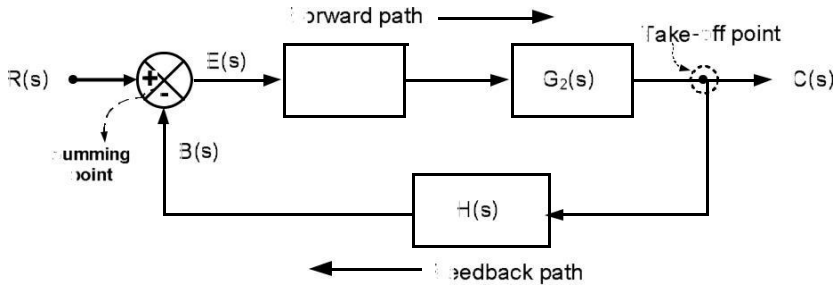


Fig.1.102. A block diagram representation of a systemshowing its different components

Take-off point: It is the component of a block diagram model at which a signal can be taken directly and supplied to one or more points as shown in Fig.1.102.

Forward path: It is the direction of signal flow from input towards output.

Feedback path: It is the direction of signal flow from output towards input.

1.3.6Developing Block Diagram model from mathematical model:

Let's discuss this concept with the following example.

Example: A system is described by following mathematical equations. Find its corresponding block diagram model.

$$x_1 \quad 3x_1 \quad 2x_2 \quad 5x_3 \tag{1.103}$$

$$x_2 \quad x_1 \quad 4x_2 \quad 3x_3 \tag{1.104}$$

$$x_3 \; 2x_1 \quad x_2 \quad x_3 \tag{1.105}$$

Example: Eq (1.103), (1.104) and (1.105) are combiningly results in the following block diagram model.

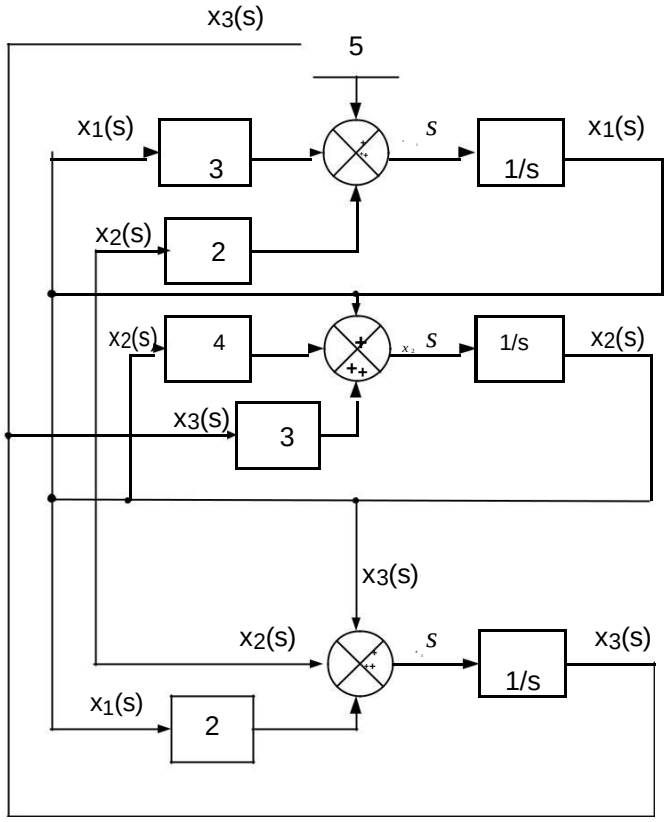
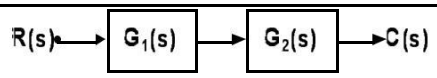
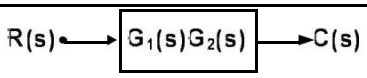
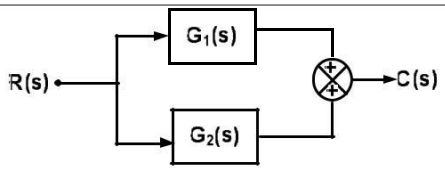
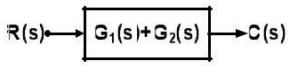
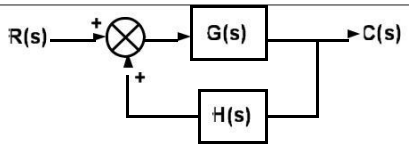
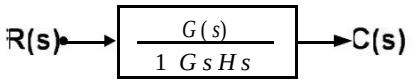
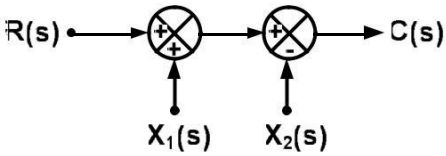
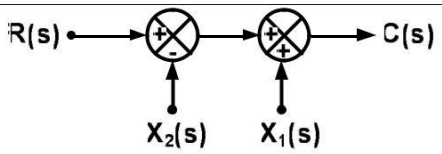
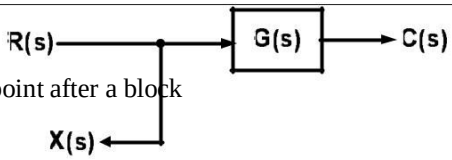
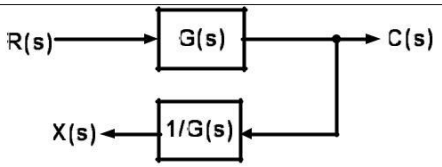
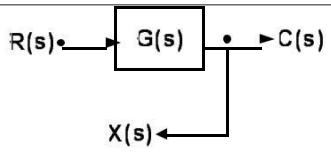
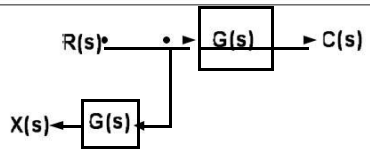
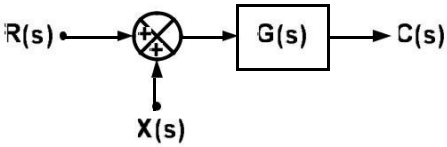
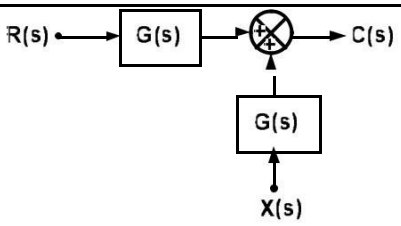
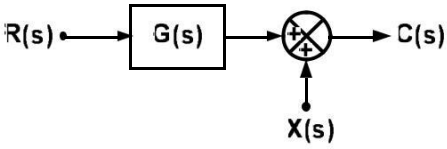
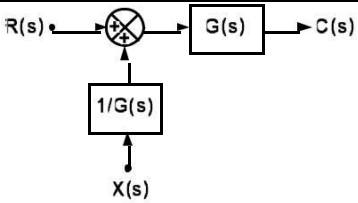


Fig.1.103. A block diagram representation of the above example

1.103. Rules for reduction of Block Diagram model:

Sl. No.	Rule No.	Configuration	Equivalent	Name
1	Rule 1			Cascade
2	Rule 2			Parallel
3	Rule 3			Loop
4	Rule 4			Associative Law
	Rule 5			Move take-off point after a block
6	Rule 6			Move take-off point before a block
7	Rule 7			Move summing-point point after a block
8	Rule 8			Move summing-point point before a block

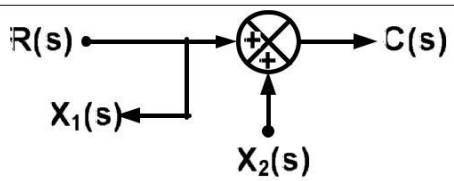
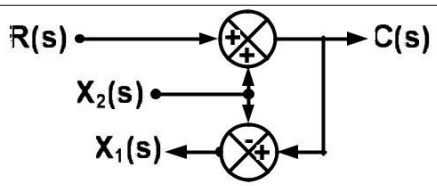
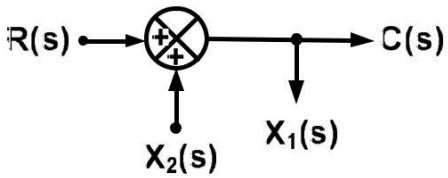
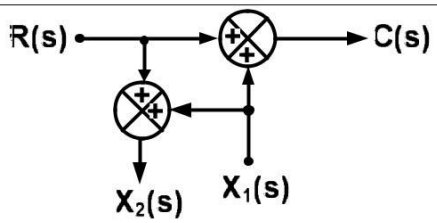
9	**Rule 9**			Move take-off point after a summing-point
10	**Rule 10**			Move take-off point before a summing-point

Fig.1.104. Rules for reduction of Block Diagram model

1.3.6Procedure for reduction of Block Diagram model:

Step 1: Reduce the cascade blocks.

Step 2: Reduce the parallel blocks.

Step 3: Reduce the internal feedback loops.

Step 4: Shift take-off points towards right and summing points towards left.

Step 5: Repeat step 1 to step 4 until the simple form is obtained.

Step 6: Find transfer function of whole system as $\frac{C\,s}{R\,s}$.

1.3.6Procedure for finding output of Block Diagram model with multiple inputs:

Step 1: Consider one input taking rest of the inputs zero, find output using the procedure described in section 1.2.

Step 2: Follow step 1 for each inputs of the given Block Diagram model and find their corresponding outputs.

Step 3: Find the resultant output by adding all individual outputs.

Signal Flow Graphs (SFGs)

It is a pictorial representation of a system that graphically displays the signal transmission in it.

1.11.1. Basic Definitions in SFGs:

Input or source node: It is a node that has only outgoing branches i.e. node 'r' in Fig.1.11.1.

Output or sink node: It is a node that has only incoming branches i.e. node 'c' in Fig.1.11.1.

Chain node: It is a node that has both incoming and outgoing branches i.e. nodes 'x_1', 'x_2','x_3','x_4','x_5'and 'x_6' in Fig.1.11.1.

Gain or transmittance: It is the relationship between variables denoted by two nodes or value of a branch. In Fig.1.11.1, transmittances are 't_1', 't_2','t_3','t_4','t_5'and 't_6'.

Forward path: It is a path from input node to output node without repeating any of the nodes in between them. In Fig.1.11.1, there are two forward paths, i.e. path-1:'r-x_1-x_2-x_3-x_4-x_5-x_6-c' and path-2:'r-x_1-x_3-x_4-x_5-x_6-c'.

Feedback path: It is a path from output node or a node near output node to a node near input node without repeating any of the nodes in between them (Fig.1.11.1).

Loop: It is a closed path that starts from one node and reaches the same node after trading through other nodes. In Fig.1.11.1, there are four loops, i.e. loop-1:'x_2-x_3-x_4-x_1', loop-2:'x_5-x_6-x_5', loop-3:'x_1-x_2-x_3-x_4-x_5-x_6-x_1' and loop-4:'x_1-x_3-x_4-x_5-x_6-x_1'.

Self Loop: It is a loop that starts from one node and reaches the same node without trading through other nodes i.e. loop in node 'x_4' with transmittance 't_{55}' in Fig.1.11.1.

Path gain: It is the product of gains or transmittances of all branches of a forward path. In Fig.1.11.1, the path gains are $P_1 = t_1t_2t_3t_4t_5$ (for path-1) and $P_2 = t_9t_3t_4t_5$ (for path-2).

Loop gain: It is the product of gains or transmittances of all branches of a loop In Fig.1.11.1, there are four loops, i.e. $L_1 = -t_2t_3t_6$, $L_2 = -t_5t_7$, $L_3 = -t_1t_2t_3t_4t_5t_8$, and $L_4 = -t_9t_3t_4t_5t_2$.

Dummy node: If the first node is not an input node and/or the last node is not an output node than a node is connected before the existing first node and a node is connected after the existing last node with unity transmittances. These nodes are called dummy nodes. In Fig.1.11.1, 'r' and 'c' are the dummy nodes.

Non-touching Loops: Two or more loops are non-touching loops if they don't have any common nodes between them. In Fig.1.11.1, L_1 and L_2 are non-touching loops

Example:

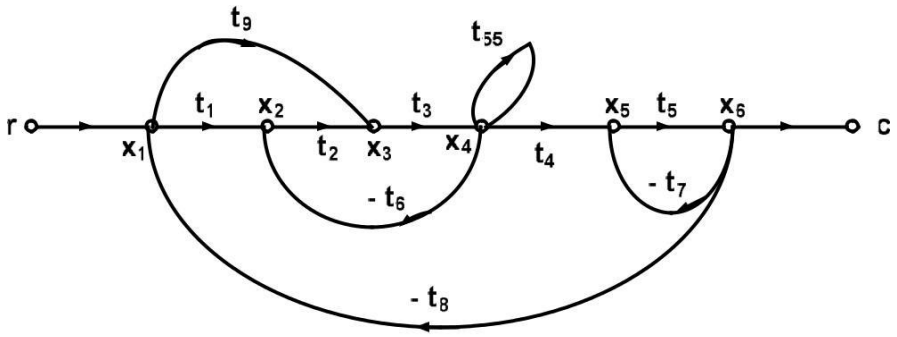


Fig.1.11.1. Example of a SFG model

1.11.2. Properties SFGs:

Applied to linear system
Arrow indicates signal flow
Nodes represent variables, summing points and take-off points
Algebraic sum of all incoming signals and outgoing nodes is zero SFG of a system is not unique
Overall gain of an SFG can be determined by using Mason's gain formula

1.11.3. SFG from block diagram model:

Let's find the SFG of following block diagram model shown in Fig.1.11.2.

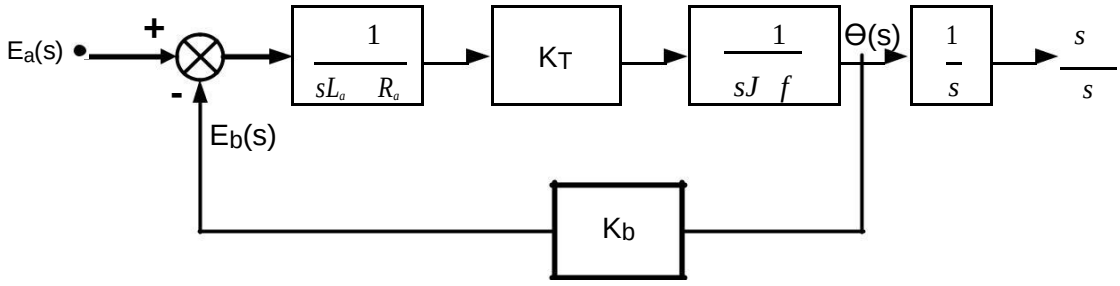


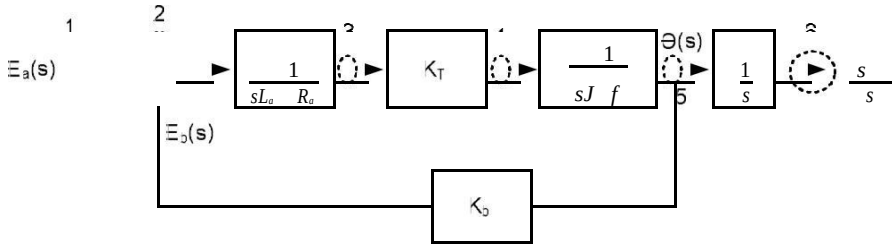
Fig.1.11.2. Armature type speed control of a DC motor

Step-1: All variables and signals are replaced by nodes.

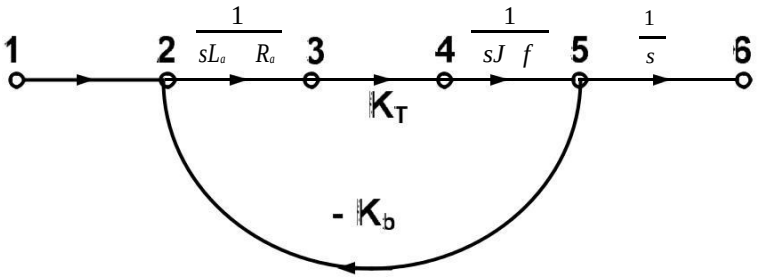
Step-2: Connect all nodes according to their signal flow.

Step-3: Each ofgains is replaced by transmittances of the branches connected between two nodes of the forward paths.

Step-4: Each ofgains is replaced by transmittances multiplied with (-1) of the branches connected between two nodes of the forward paths.



(a)



(b)

Fig.1.11.3. Armature type speed control of a DC motor

1.11.4. Mason's gain formula:

Transfer function of a system=

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_{k=1}^{N} P_k \Delta_k}{\Delta} \qquad (1.11.1)$$

Where,

N= total number of forward paths

P_k= path gain of k^{th} forward path

Δ= 1 - (∑loop gains of all individual loops) + (∑gain product of loop gains of all possible two non-touching loops) - (∑gain product of loop gains of all possible three non-touching loops) + …

Δ_k= value of Δ after eliminating all loops that touches k^{th} forward path

Example:

Find the overall transfer function of the system given in Fig.1.11.1 using Mason's gain formula.

Solution:

In Fig.1.11.1,

No. of forward paths: $N = 2$

Path gain of forward paths: $P_1 = t_1 t_2 t_3 t_4 t_5$ and $P_2 = t_6 t_3 t_4 t_5$

Loop gain of individual loops: $L_1 = t_2 t_3 t_6$, $L_2 = t_5 t_7$, $L_3 = t_1 t_2 t_3 t_4 t_5 t_8$ and $L_4 = t_9 t_3 t_4 t_5 t_8$

No. of two non-touching loops = 2 i.e. L_1 and L_2

No. of more than two non-touching loops = 0

$1\ L_1\ L_2\ L_3\ L_4\ L_1\ L_2\ 0\ 1\ L_1\ L_2\ L_3\ L_4\ L_1\ L_2\ _1\ 1\ 0\ 1$ and $_2\ 1\ 0\ 1$

$$G\ s\quad \frac{P_{1\ 1}\quad P_{2\ 2}}{}$$

$$G\ s\quad \frac{t_1 t_2\ t_3\ t_4\ t_5\qquad 1\quad t_6\ t_3\ t_4\ t_5\qquad 1}{1\quad t_2\ t_3\ t_6\quad t_5\ t_7\quad t_1 t_2\ t_3\ t_4\ t_5\ t_8\quad t_9\ t_3\ t_4\ t_5\ t_8\quad {}_{t_2 t_3 t_5 t_6 t_7}}$$

$$G\quad s\quad \frac{{}_{t_1 t_2 t_3 t_4 t_5}\quad {}_{t_6 t_3 t_4 t_5}}{1\quad t_2\ t_3\ t_6\quad t_5\ t_7\quad t_1 t_2\ t_3\ t_4\ t_5\ t_8 t_9\ t_3\ t_4\ t_5\ t_8\quad {}_{t_2 t_3 t_5 t_6 t_7}}$$

Feedback Characteristics of Control System

1.12.1. Feedback and Non-feedback Control systems

Non-feedback control system: It is a control system that does not have any feedback paths. It is also known as open-loop control system. It is shown in Fig.1.12.1 (a) and (b).

Feedback control system: It is a control system that has at least one feedback path. It is also known as closed-loop control system. It is shown in Fig.1.12.2 (a) and (b).

Fig.1.12.1. (a) Block diagram of a non-feedback control system and (b) SFG of a non-feedback control system

Fig.1.12.2. (a) Block diagram of a feedback control system and (b) SFG of a feedback control system

1.12.2. Types of Feedback in a Control system

1.12.2.1. Degenerative feedback control system: It is a control system where the feedback signal opposes the input signal. Here,

Error or actuating signal = (Input signal) – (Feedback signal).

Referring Fig.1.12.3,

$$E\ s\quad R\ s\quad B\ s \qquad (1.12.1)$$

and

$$T_1 s \quad \frac{G s}{1 \; G s H \; s} \tag{1.12.2}$$

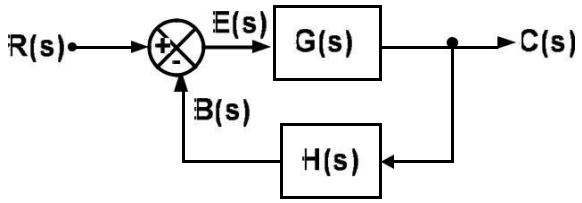


Fig.1.12.3. (a) Block diagram of a degenerative feedback control system

1.12.2.2. Regenerative feedback control system: It is a control system where the feedback signal supports or adds the input signal. Here,

Error or actuating signal = (Input signal) + (Feedback signal).

Referring Fig.1.12.4,

$$E\,s \quad R\,s \quad B\,s \tag{1.12.3}$$

and

$$T_2\ s \quad \frac{G\,s}{1 \quad G\,s\,H\,s} \tag{1.12.4}$$

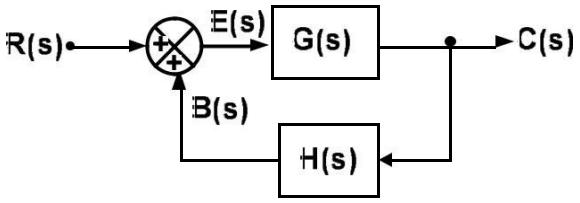


Fig.1.12.4. Block diagram of a regenerative feedback control system

1.12.3. Effect of parameter variation on overall gain of a degenerative Feedback Control system

The overall gain or transfer function of a degenerative feedback control system depends upon these parameters i.e. (i) variation in parameters of plant, and (ii) variation in parameter of feedback system and (ii) disturbance signals.

The term sensitivity is a measure of the effectiveness of feedback on reducing the influence of any of the above described parameters. For an example, it is used to describe the relative variations in the overall Transfer function of a system T(s) due to variation in G(s).

$$= \frac{\hbar \qquad ()}{\hbar}$$

1.12.3.1. Effect of variation in G(s) on T(s) of a degenerative Feedback Control system

In an open-loop system,

$$C\,s \quad G\,s\,R\,s$$

Let, due to parameter variation in plant G(s) changes to [G(s) + ΔG(s)] such that |G(s)| >> |ΔG(s)|. The output of the open-loop system then changes to

$$C\,s \quad C\,s \quad G\,s \quad G\,s \quad R\,s$$

$$C\,sC\,s \quad G\,s\,R\,sG\,s\,R\,s$$

$$C\,sG\,s\,R\,s \tag{1.12.5}$$

In an closed-loop system,

$$C(s) = \frac{G(s)}{1 + G(s)H(s)} R(s)$$

Let, due to parameter variation in plant G(s) changes to [G(s) + ΔG(s)] such that |G(s)| >> |ΔG(s)|. The output of the open-loop system then changes to

$$C + \Delta C = \frac{(G + \Delta G)}{1 + (G + \Delta G)H} R$$

$$C + \Delta C = \frac{(G + \Delta G)}{1 + GH + \Delta GH} R$$

Since, |G(s)| >> |ΔG(s)|, then $G(s)H(s) \gg \Delta G(s)H(s)$. Therefore, $\Delta G(s)H(s)$ is neglected. Now,

$$C + \Delta C = \frac{(G + \Delta G)}{1 + GH} R$$

$$C + \Delta C = \frac{G}{1 + GH} R + \frac{\Delta G}{1 + GH} R$$

Or

$$C(s) = \frac{G(s)}{1 + G(s)H(s)} R(s) \tag{1.12.6}$$

Comparing eq (42 and (43), it is clear that Δ ( ) = (1 +) Δ ( )

This concept can be reproved using sensitivity. Sensitivity on T(s) due to variation in G(s) is given by

$$S_G^T = \frac{T'}{G'} \frac{G}{T} \tag{1.12.7}$$

For open-loop system,

$$S_G^T = \frac{T'}{G'} \frac{G}{T} = \frac{G}{G} \frac{G}{G} = 1 \tag{1.12.8}$$

For closed-loop system,

$$S_G^T = \frac{T'}{G'} \frac{G}{T} = \frac{1 + GH}{1 + GH^2} \frac{G}{1 + GH} = \frac{1}{1 + GH} \tag{1.12.9}$$

Therefore, it is proved that ( ) = (1 +) ( ). Hence, the effect of parameter variation in case of closed loop system is reduced by a factor of ( ) .

1.12.3.2. Effect of variation in H(s) on T(s) of a degenerative Feedback Control system

This concept can be reproved using sensitivity. Sensitivity on T(s) due to variation in H(s) is given by

$$S_H^T \frac{T}{H}\frac{T}{H} \quad \frac{T}{H} \quad \frac{H}{T} \tag{1.12.10}$$

For closed-loop system,

$$S_H^T \quad \frac{T}{H}\frac{H}{T} \quad G\frac{G}{1 GH^2} \quad G \frac{H}{1/GH} \quad \frac{GH}{1 GH} \tag{1.12.11}$$

For higher value of GH, sensitivity approaches unity. Therefore, change in H affects directly the system output.

2. Time Domain Analysis of Control Systems

2.1. Time response

Time response c(t)is the variation of output with respect to time. The part of time response that goes to zero after large interval of time is called transient response $c_{tr}(t)$. The part of time response that remains after transient response is called steady-state response $c_{ss}(t)$.

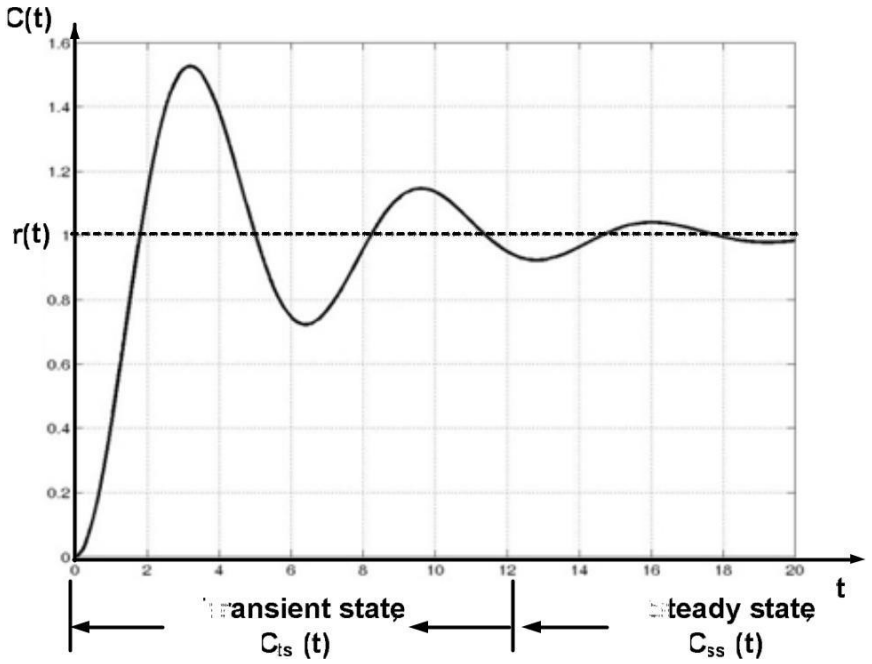


Fig.1.12.1. Time response of a system

2.2. System dynamics

System dynamics is the study of characteristic and behaviour of dynamic systems

i.e.

i. Differential equations: First-order systems and Second-order systems,
ii. Laplace transforms,
iii. System transfer function,
iv. Transient response: Unit impulse, Step and Ramp

Laplace transforms convert differential equations into algebraic equations. They are related to frequency response

$$\mathbf{L}\,\mathbf{x}\,\mathbf{t} \quad \mathbf{X}\,\mathbf{s} x(t) e^{st} dt \qquad (2.1)$$

$$0$$

No.	Function	Time-domain $x(t) = \mathcal{L}^{-1}\{X(s)\}$	Laplace domain $X(s) = \mathcal{L}\{x(t)\}$
1	Delay	$\delta(t-\tau)$	$e^{-\tau s}$
2	Unit impulse	$\delta(t)$	1
3	Unit step	$u(t)$	$\frac{1}{s}$
4	Ramp	t	$\frac{1}{s^2}$
5	Exponential decay	$e^{-\alpha t}$	$\frac{1}{s}$
6	Exponential approach	$1 \quad e^{\quad t}$	$\frac{\quad}{s(s \quad)}$
7	Sine	$\sin \omega t$	$\frac{\quad}{s^2 \quad ^2}$
8	Cosine	$\cos \omega t$	$\frac{s}{s^2 \quad ^2}$
9	Hyperbolic sine	$\sinh \alpha t$	$\frac{\quad}{s^{2\ 2}}$
10	Hyperbolic cosine	$\cosh \alpha t$	$\frac{s}{s^2 \quad ^2}$
11	Exponentially decaying sine wave	$e^{\quad t} \sin \quad t$	$\frac{\quad}{(s \quad)^2 \quad ^2}$
12	Exponentially decaying cosine wave	$e^{\quad t} \cos \quad t$	$\frac{s}{(s \quad)^2 \quad ^2}$

2.3. Forced response

$$\frac{C(s)}{z_m)\ R(s)} \qquad \frac{G(s)R(s)}{} \qquad \frac{K(s \quad z_1)(s \quad z_2)(s}{(s \quad p_1)(s \quad p_2)\ (s \quad p_n)} \tag{2.2}$$

R(s) input excitation

2.4. Standard test signals

2.1.1.11. Impulse Signal: An impulse signal δ(t) is mathematically defined as follows.

$$\begin{matrix} & undefined & ; t \quad 0 \\ t & & \\ & 0 & ; t \quad 0 \end{matrix} \tag{2.3}$$

Laplace transform of impulse signal is

$$s \quad 1 \tag{2.4}$$

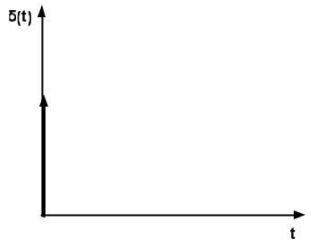
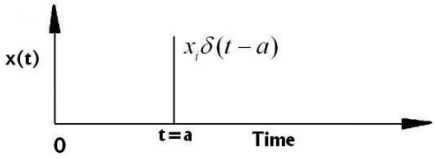


Fig.1.12.2. Impulse signal

Dirac delta function

$$x(t) \; x_i \; (t \quad a) \tag{2.5}$$



Integral property of Dirac delta function

$$(t)\,(\,t \qquad t_o\,)\,dt \quad (t_o) \tag{2.6}$$

Laplace transform of an impulse input

$$X(s) \quad e^{st} x_i \; (t \quad a)dt \quad x_i e^{sa} \tag{2.7}$$

0

2.1.1.12. Step Signal: A step signal u(t) is mathematically defined as follows.

$$u\,t \quad \begin{cases} 0 & ;\, t \quad 0 \\ K & ;\, t \qquad 0 \end{cases} \tag{2.8}$$

Laplace transform of step signal is

$$U\ s = \frac{K}{s} \tag{2.9}$$

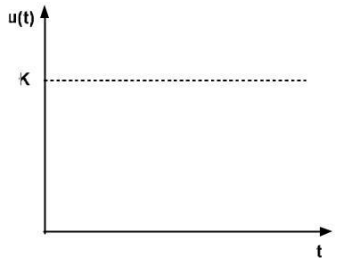


Fig.1.12.2. Step signal

2.1.2. Ramp Signal: A step signal r(t) is mathematically defined as follows.

$$r\,t \quad \begin{matrix} 0 & ;\, t\ 0 \\ Kt & ;\, \mathrm{t}\ 0 \end{matrix} \tag{2.10}$$

Laplace transform of ramp signal is

$$R\,s \quad \frac{K}{s^2} \tag{2.11}$$

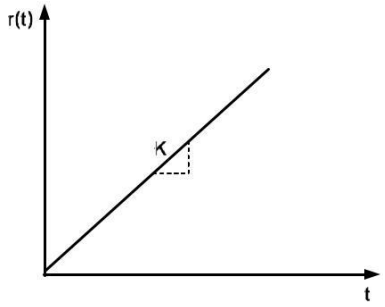


Fig.1.12.3. Ramp signal

2.4.4. Parabolic Signal A step signal a(t) is mathematically defined as follows.

$$a\,t \quad \begin{matrix} 0 & ;\, t\ 0 \\ \dfrac{Kt^{\,2}}{2} & ;\, \mathrm{t}\ 0 \end{matrix} \tag{2.12}$$

Laplace transform of parabolic signal is

$$A\,s \quad \frac{K}{s^3} \tag{2.13}$$

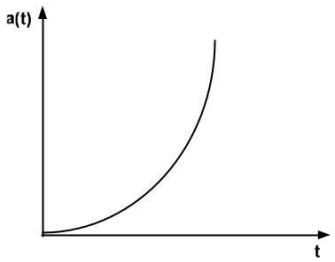


Fig.1.12.4. Parabolic signal

2.4.1.10 Sinusoidal Signal A sinusoidal x(t) is mathematically defined as follows.

$$x\,t \quad \sin\ t \tag{2.14}$$

Laplace transform of sinusoidal signal is

$$X\,se^{\,st}\sin\ t\,dt \qquad \frac{}{s^{\,2}\quad 2} \tag{2.15}$$

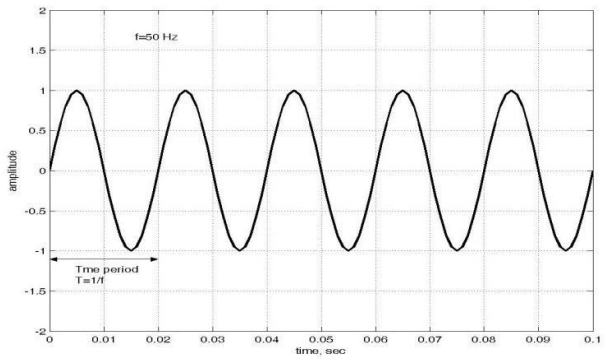


Fig.1.12.4. Sinusoidal signal

2.1.10 Steady-state error:

A simple closed-loop control system with negative feedback is shown as follows.

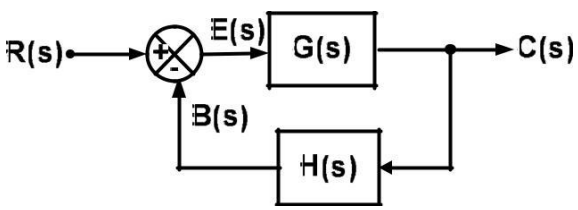


Fig.1.12.1.10 A simple closed-loop control system with negative feedback

Here,

$$E\,s \quad R\,s \quad B\,s \tag{2.16}$$

$$B\,s \quad C\,s\,H\,s \tag{2.17}$$

$$C\,s \quad E\,s\,G\,s \tag{2.18}$$

Applying (1) in (9),

$$E\,s \quad R\,s \quad C\,s\,H\,s \tag{2.19}$$

Using (11) in (12),

$$E\,s \quad R\,s \quad E\,s\,G\,s\,H\,s \tag{2.20}$$

$$1\ G\,s\,H\,s\ E\,s\ R\,s \tag{2.21}$$

$$E(s) = \frac{R(s)}{1 + G(s)H(s)} \tag{2.22}$$

Steady-state error,

$$e_{ss} = \lim_{t \to} e(t) = \lim_{s \to 0} sE(s) \tag{2.23}$$

Using (15) in (16),

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)} \tag{2.24}$$

Therefore, steady-state error depends on two factors, i.e.

(a) type and magnitude of R(s)
(b) open-loop transfer function G(s)H(s)

2.1.11. Types of input and Steady-state error:

2.1.11.1. Step Input (2.25)

$$R(s) = \frac{A}{s}$$

Using (18) in (17),

(2.26)

$$e_{ss} = \lim_{s \to 0} \frac{s\frac{A}{s}}{1 + G(s)H(s)} = \lim_{s \to 0} \frac{A}{1 + G(s)H(s)}$$

(2.27)

$$e_{ss} = \frac{A}{1 + \lim_{s \to 0} G(s)H(s)} = \frac{A}{1 + K_P}$$

Where, (2.28)

$$K_P = \lim_{s \to 0} G(s)H(s)$$

2.1.11.2. Ramp Input (2.29)

$$R(s) = \frac{A}{s^2}$$

Using (18) in (17),

$$e_{ss} = \lim_{s \to 0} \frac{s\frac{A}{s^2}}{1 + G(s)H(s)} = \lim_{s \to 0} \frac{A}{s[1 + G(s)H(s)]} \tag{2.30}$$

$$e_{ss} = \lim_{s \to 0} \frac{A}{s + sG(s)H(s)}$$

$$e_{ss} = \frac{A}{\lim_{s \to 0} sG(s)H(s)} = \frac{A}{K_V}$$

Where,

$$K_V = \lim_{s \to 0} sG(s)H(s) \tag{2.31}$$

2.1.11.3. Parabolic Input

$$R(s) = \frac{A}{s^3} \tag{2.32}$$

Using (18) in (17),

$$e_{ss} = \lim_{s \to 0} \frac{\frac{A}{s^3}}{1 + G(s)H(s)} = \lim_{s \to 0} \frac{A}{s^2 + s^2 G(s)H(s)}$$

$$e_{ss} = \lim_{s \to 0} \frac{A}{s^2 + s^2 G(s)H(s)} \tag{2.33}$$

$$e_{ss} = \frac{A}{\lim_{s \to 0} s^2 G(s)H(s)} = \frac{A}{K_A}$$

Where,

$$K_A = \lim_{s \to 0} s^2 G(s)H(s) \tag{2.34}$$

Types of input and steady-state error are summarized as follows.

Error Constant	**Equation**	**Steady-state error (e_{ss})**
Position Error Constant (K_P)	$K_P = \lim_{s \to 0} G(s)H(s)$	$e_{ss} = \frac{A}{1 + K_P}$
Velocity Error Constant (K_V)	$K_V = \lim_{s \to 0} sG(s)H(s)$	$e_{ss} = \frac{A}{K_V}$
Acceleration Error Constant (K_A)	$K_A = \lim_{s \to 0} s^2 G(s)H(s)$	$e_{ss} = \frac{A}{K_A}$

2.1.12. Types of open-loop transfer function G(s)H(s)and Steady-state error:

2.1.12.1. Static Error coefficient Method

The general form of G(s)H(s) is

$$G(s)H(s) = \frac{K(1 + T_1 s)(1 + T_2 s)\ldots(1 + T_n s)}{s^j (1 + T_a s)(1 + T_b s)\ldots(1 + T_m s)} \tag{2.35}$$

Here, j = no. of poles at origin (s = 0)

or, type of the system given by eq (28) is j.

2.1.12.1.1. Type 0

$$G(s)H(s) = \frac{K(1 + T_1 s)(1 + T_2 s)\ldots(1 + T_n s)}{(1 + T_a s)(1 + T_b s)\ldots(1 + T_m s)} \tag{2.36}$$

Here,

$$K_P = \lim_{s \to 0} G(s)H(s) = K \tag{2.37}$$

Therefore,

$$e_{ss} = \frac{A}{1 + K} \tag{2.38}$$

2.1.12.1.2. Type 1

$$G(s)H(s) = \frac{K(1 + T_1 s)(1 + T_2 s)\ldots(1 + T_n s)}{s(1 + T_a s)(1 + T_b s)\ldots(1 + T_m s)} \tag{2.39}$$

Here,

$$K_V = \lim_{s \to 0} sG(s)H(s) = K \tag{2.40}$$

Therefore,

$$e_{ss} = \frac{A}{K} \tag{2.41}$$

2.1.12.1.3. Type 2

$$G(s)H(s) = \frac{K(1 + T_1 s)(1 + T_2 s)\ldots(1 + T_n s)}{s^2(1 + T_a s)(1 + T_b s)\ldots(1 + T_m s)} \tag{2.42}$$

Here,

$$K_A = \lim_{s \to 0} s^2 G(s)H(s) = K \tag{2.43}$$

Therefore,

$$e_{ss} = \frac{A}{K} \tag{2.44}$$

Steady-state error and error constant for different types of input are summarized as follows.

Type	**Step input**		**Ramp input**		**Parabolic input**	
	K_P	e_{ss}	K_V	e_{ss}	K_A	e_{ss}
Type 0	K	$\frac{A}{1 + K}$	0		0	
Type 1		0	K	$\frac{A}{K}$	0	
Type 2		0		0	K	$\frac{A}{K}$

The static error coefficient method has following advantages:

Can provide time variation of error Simple calculation

But, the static error coefficient method has following demerits:

Applicable only to stable system
Applicable only to three standard input signals
Cannot give exact value of error. It gives only mathematical value i.e. 0 or ∞

2.1.12.2. Generalized Error coefficient Method

From eq (15),

$$E(s) = \frac{1}{1 + G(s)H(s)} R(s)$$

So,

$$E(s) = F_1(s) F_2(s) \tag{2.45}$$

Where, $F_1 = \frac{1}{1 + G(s)H(s)}$ and $F_2(s) = R(s)$

Using convolution integral to eq (38)

$$e(t) = f_1(t) * f_2(t) = \int_0^t f_2(t) f_1(t - \tau) d\tau = \int_0^t r(\tau) d\tau \tag{2.46}$$

Using Taylor's series of expansion to $r(t)$,

$$r(t) = r(0) + \dot{r}(0)t + \frac{\ddot{r}(0)}{2!}t^2 + \frac{\dddot{r}(0)}{3!}t^3 + \dots \tag{2.47}$$

Now, applying eq (40) in eq (39),

$$e(t) = \int_0^t f_1(\tau) r(t - \tau) d\tau = r(t) * f_1(t) = \int_0^t r(\tau) f_1(t - \tau) d\tau + \int_0^t \frac{\ddot{r}(0)}{2!} f_1(t - \tau) d\tau + \int_0^t \frac{\dddot{r}(0)}{3!} f_1(t - \tau) d\tau + \dots \tag{2.48}$$

Now, steady-state error, e_{ss} is

$$e_{ss} = \lim_{t \to \infty} e(t) \tag{2.49}$$

Therefore,

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{t \to \infty} \int_0^t r(t - \tau) f_1(\tau) d\tau + \int_0^t r(t - \tau) f_1(\tau) d\tau + \int_0^t \frac{\ddot{r}(0)}{2!} f_1(\tau) d\tau + \int_0^t \frac{\dddot{r}(0)}{3!} f_1(\tau) d\tau + \dots$$

$$e_{ss} = \int_0^\infty f_1(\tau) d\tau \int_0^t r(t - \tau) d\tau + \int_0^\infty f_1(\tau) d\tau \int_0^t \frac{\ddot{r}(0)}{2!} r(t - \tau) d\tau + \int_0^\infty f_1(\tau) d\tau \int_0^t \frac{\dddot{r}(0)}{3!} r(t - \tau) d\tau + \dots \tag{2.50}$$

Eq (44) can be rewritten as

$$e_{ss} = C_0 r(t) + C_1 \dot{r}(t) + \frac{C_2}{2!} \ddot{r}(t) + \frac{C_3}{3!} \dddot{r}(t) + \dots \tag{2.51}$$

Where, C_0, C_1, C_2, C_3, etc. are dynamic error coefficients. These are given as

$$C_0 = f_1(0) \quad d = \lim_{s \to 0} F_1(s)$$

$$C_1 = f_1'(0) \quad d = \lim_{s \to 0} \frac{dF_1(s)}{ds}$$

$$C_2 = \frac{1}{2!} f_1''(0) \quad d = \lim_{s \to 0} \frac{d^2 F(s)}{ds^2}, \text{ and so on…} \qquad (2.52)$$

$$C_3 = \frac{1}{3!} f_1'''(0) \quad d = \lim_{s \to 0} \frac{d^3 F(s)}{ds^3}$$

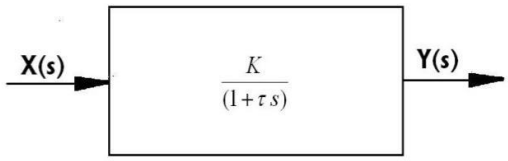
2.2. First-order system:

A Governing differential equation is given by

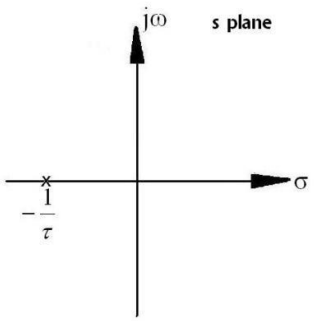
$$y + y = Kx(t) \qquad (2.53)$$

Where, Time constant, sec = ,
Static sensitivity (units depend on the input and output variables) = K ,
y(t) is response of the system and
x(t) is input excitation
The System transfer function is

$$\frac{Y(s)}{X(s)} = G(s) = \frac{K}{(1 + s)} \qquad (2.54)$$



Pole-zero map of a first-order system



Normalized response

In this type of response

Static components are taken out leaving only the dynamic component

The dynamic components converge to the same value for different physical systems of the same type or order

Helps in recognizing typical factors of a system

2.2.1. Impulse input to a first-order system

Governing differential equation

$$y \quad y \quad Kx_i \ (t) \tag{2.55}$$

Laplacian of the response

$$Y(s) \quad \frac{Kx_i}{(1 \quad s)} \quad \frac{Kx_i}{s} \quad \frac{1}{1} \tag{2.56}$$

Time-domain response

$$y(t) \quad \frac{Kx_i}{} \quad e^{\frac{t}{}} \tag{2.57}$$

Impulse response function of a first-order system

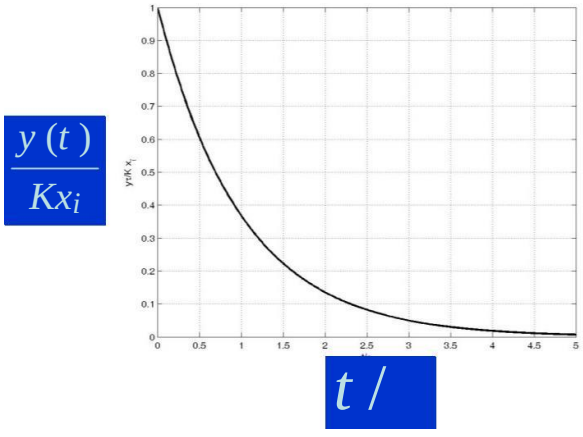
$$h(t) \frac{K}{} \quad e^{\frac{t}{}} \tag{2.58}$$

By putting $x_i = 1$ in the response

Response of a first-order system to any force excitation

$$y(t) \frac{K}{} \int_0^t e^{\frac{t}{}} F(t \quad) \, d \tag{2.59}$$

The above equation is called Duhamel's integral. Normalized response of a first-order system to impulse input is shown below.



2.2.2. Step input to a first-order system

Governing differential equation

$$y \quad y \quad Kx_i \, u \ (t) \tag{2.60}$$

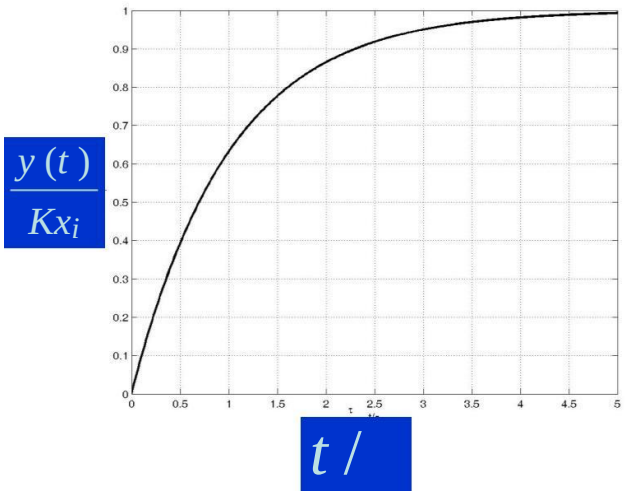
Laplacian of the response

$$Y(s) \quad \frac{Kx_i}{s(1 \quad s)} \quad \frac{Kx_i}{s} \quad \frac{Kx_i}{s \quad \frac{1}{}} \tag{2.61}$$

Time-domain response

$$y(t) \quad Kx_i \quad 1 \quad e^{\frac{t}{}} \tag{2.62}$$

Normalized response of a first-order system to impulse input is shown below.



2.2.3. Ramp input to a first-order system

Governing differential equation

$$y \quad y \quad Kt \tag{2.63}$$

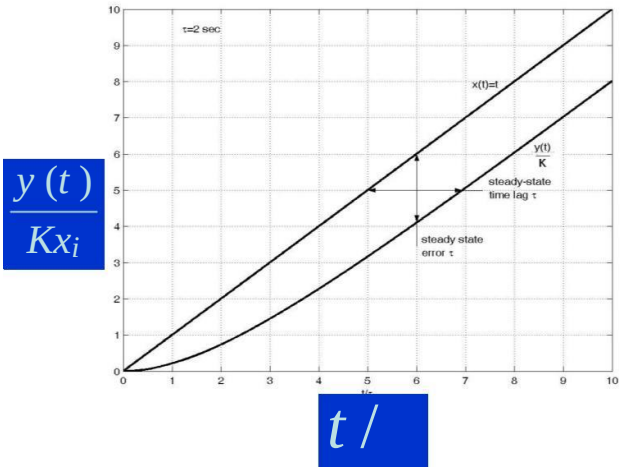
Laplacian of the response

$$Y(s) \quad \frac{K}{s^2(1 \quad s)} \quad \frac{1}{s^2} \quad \frac{}{s} \quad \frac{}{s} \quad \frac{}{1} \tag{2.64}$$

Time-domain response

$$\frac{y(t)}{K} \qquad te^{\frac{t}{}} \tag{2.65}$$

Normalized response of a first-order system to impulse input is shown below.



2.2.4. Sinusoidal input to a first-order system

Governing differential equation

$$y \quad y \quad KA\sin \quad t \tag{2.66}$$

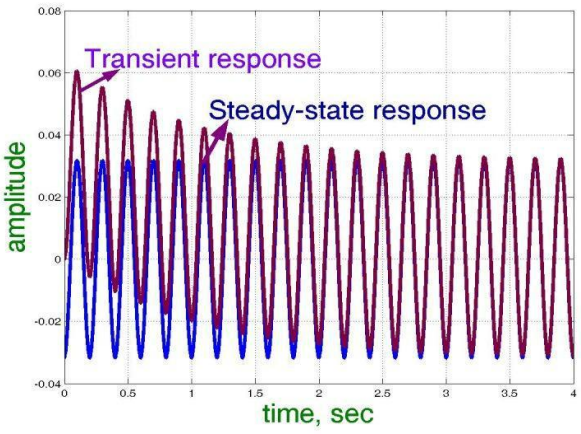
Laplacian of the response

$$Y(s) \quad \frac{K \quad A}{(1 \quad s)\,s} \quad \frac{}{1} \quad \frac{}{s \quad 1/} \quad \frac{s}{s} \quad \frac{1}{s} \tag{2.67}$$

Time-domain response

$$\frac{y(t)}{KA} \quad \frac{}{1^2} e^{\quad t/} \quad \cos \quad t \quad \frac{1}{} \quad \sin \quad t \tag{2.68}$$

Normalized response of a first-order system to impulse input is shown below.



2.9. Second-order system

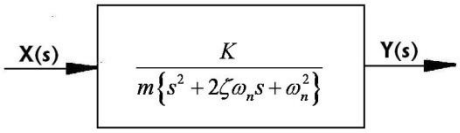
A Governing differential equation is given by

$$my\ cy\ \ ky\ \ Kx\ (t) \tag{2.69}$$

Where, = Time constant, sec,
K = Static sensitivity (units depend on the input and output variables), m = Mass (kg),
c = Damping coefficient (N-s/m), k = Stiffness (N/m),
y(t) is response of the system
and x(t) is input excitation

The System transfer function is

$$\frac{Y(s)}{X(s)} \quad \frac{K}{m\,s^{2}\ \ 2\ _{n}\,s\ \ _{n}^{2}} \tag{2.70}$$

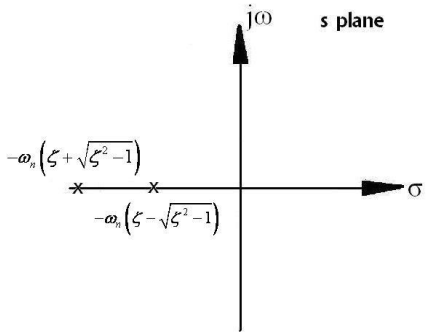


Pole-zero map

(a) ζ>1 over damped
Poles are:

$$s_{1,2n} \qquad \sqrt{^{2}\ \ 1} \tag{2.71}$$

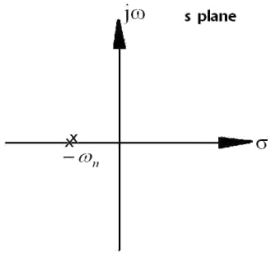
Graphically, the poles of an over damped system is shown as follows.



(b) ζ =1 critically
damped Poles are:

$$s_{1,2n} \tag{2.72}$$

Graphically, the poles of an critically damped system is shown as follows.



(c) ζ<1 under damped
Poles are:

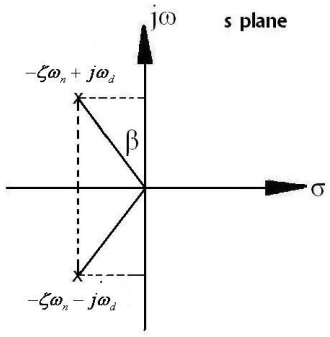
$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \qquad (2.73)$$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_d$$

Where, ω_d Damped natural frequency

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \qquad (2.74)$$

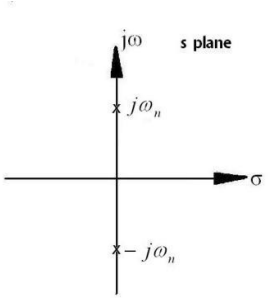
Graphically, the poles of an critically damped system is shown as follows.



Here, $\tan\beta = \frac{\sqrt{1-\zeta^2}}{\zeta}$

(d) ζ = 0 un-damped
Poles are:

$$s_{1,2} = \pm j\omega_n \qquad (2.75)$$



Solved problems:

1. A single degree of freedom spring-mass-damper system has the following data: spring stiffness 20 kN/m; mass 0.05 kg; damping coefficient 20 N-s/m. Determine
   (a) undamped natural frequency in rad/s and Hz
   (b) damping factor
   (c) damped natural frequency n rad/s and Hz.

If the above system is given an initial displacement of 0.1 m, trace the phasor of the system for three cycles of free vibration.

Solution:

$$_n \quad \sqrt{\frac{k}{m}} \quad \sqrt{\frac{20\ 10^3}{0.05}} \quad 632.46 \text{ rad/s}$$

$$f_n \quad \frac{_n}{2} \quad \frac{632.46}{2} \quad 100.66 \text{ Hz}$$

$$\frac{c}{2\sqrt{km}} \quad \frac{20}{2\sqrt{20\ 10^3 \quad 0.05}} \quad 0.32$$

$$_d \quad _n\sqrt{1 \quad ^2} \quad 632.46\sqrt{1 \quad 0.32^{\ 2}} \quad 600 \text{ rad/s}$$

$$f_d \quad \frac{_d}{2} \quad \frac{600}{2} \quad 91.1037 \text{ Hz}$$

$$y(t) \quad Ae^{\ _n t} \quad 0.1e^{\ 0.32\ 632.46t}$$

2. A second-order system has a damping factor of 0.3 (underdamped system) and an un-damped natural frequency of 10 rad/s. Keeping the damping factor the same, if the un-damped natural frequency is changed to 20 rad/s, locate the new poles of the system? What can you say about the response of the new system?

Solution:

Given, $_{n1}$ 10 rad/s and $_{n\,2}$ 20 rad/s

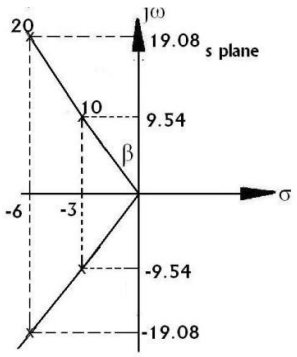
$$_{d1} \quad _{n1}\sqrt{1 \quad ^2} \quad 10\sqrt{1 \quad 0.3^2} \quad 9.54 \text{ rad/s}$$

$$_{d\,2} \quad _{n2}\sqrt{1 \quad ^2} \quad 20\sqrt{1 \quad 0.3^2} \quad 19.08 \text{ rad/s}$$

$$p_{1,2} \quad _{n1} \quad j \ _{d1} \quad 3 \quad j9.54$$

$p_{3,4} \quad {}_{n}^{2} \quad j\,{}_{d}^{2} \quad 6 \quad j19.08$

$$\tan \frac{\sqrt{1 \quad {}^{2}}}{} \quad \frac{0.3}{\sqrt{1 \quad 0.3^{2}}} \quad 11.12.4 \quad 5^{o}$$



2.9.1. Second-order Time Response Specifications with Impulse input

(a) Over damped case (ζ>1)

General equation

$$y \; 2 \; {}_{n} y \; {}_{n} \quad {}^{2} y \quad \frac{Kx_i}{m}(t) \tag{2.76}$$

Laplacian of the output

$$Y(s) \quad \frac{Kx_i}{m} \quad \frac{1}{s^{2} \; 2 \; {}_{n} s \; {}_{n}^{2}} \tag{2.77}$$

$$\frac{Kx_i}{2m^{2}\sqrt{\quad}} \; {}_{n} \; 1 \quad \frac{1}{(s \; {}_{n} \; {}_{n}\sqrt{\quad}^{2} \; 1)} \quad \frac{1}{(s \; {}_{n} \; {}_{n}\sqrt{\quad}^{2} \; 1}$$

Time-domain response

$$y(t) \frac{Kx_i}{m\sqrt{\quad}^{2} \; 1} \quad e^{{}_{n}t}\sinh \quad {}_{n}\sqrt{\quad}^{2} \; 1 \quad t \tag{2.78}$$

(b) Critically damped case (ζ=1)

General equation

$$y \quad {}_{n} \quad {}^{2} y \quad \frac{Kx_i}{m}(t) \tag{2.79}$$

Laplacian of the output

$$Y(s) \quad \frac{Kx}{m} \quad \frac{1}{s \quad {}^{2} \quad {}_{n}^{2}} \tag{2.80}$$

Time-domain response

$$y(t) \quad \frac{Kx}{m} \; {}_{n} t e \quad {}_{n}^{t} \tag{2.81}$$

(c) Under damped case (ζ<1)

Poles are: $s_{1,2} = -\zeta\omega_n \pm j\omega_d$

General equation

$$y + 2\zeta\omega_n y + \omega_n^2 y = \frac{Kx_i}{m}(t) \tag{2.82}$$

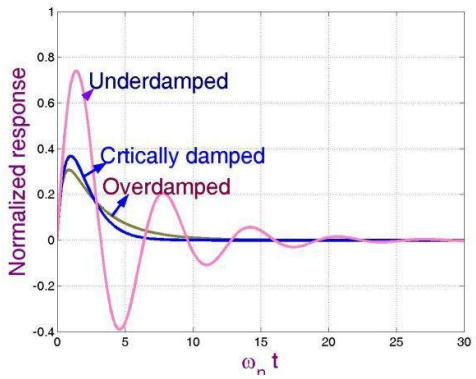
Laplacian of the output

$$Y(s) = \frac{Kx_i}{m} \cdot \frac{1}{(s + \zeta\omega_n - j\omega_d)(s + \zeta\omega_n + j\omega_d)} \tag{2.83}$$

Time-domain response

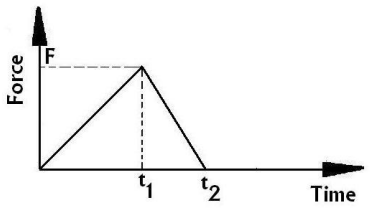
$$y(t) = \frac{Kx}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \tag{2.84}$$

Normalized impulse-response of a second-order system with different damping factors are shown graphically as follows.



Solved problems:

3. A second-order system has an un-damped natural frequency of 100 rad/s and a damping factor of 0.3. The value of the coefficient of the second time derivative (that is m) is 1.10 If the static sensitivity is 10, write down the response (do not solve) for a force excitation shown in the figure in terms of the Duhamel's integral for the following periods of time: 0<t<t1, t1<t<t2 and t>t2.



Solution:

Given, Undamped natural frequency ω_n=100 rad/s

Damping factor =0.3

Coefficient of the second time derivative m=5

Static sensitivity K=10

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 100\sqrt{1-0.3^2} = 91.1039 \text{ rad/s}$$

Here,

$$F(t) = F\frac{t}{t_1}\ ;0 \le t \le t_1$$

$$F(t) = \frac{F}{t_2 - t_1}t_2 - t\ ;t_1 \le t \le t_2$$

$$y(t) = \frac{K}{\omega_d}\int_0^t F(t)\, e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau$$

$$y(t) = \frac{10F}{5}\int_0^t 0.3\,100\, 91.1039\tau\, e^{-\zeta\omega_n(t-\tau)}\sin 91.1039(t-\tau)\,d\tau \quad ;0 \le t \le t_1,$$

$$\frac{0.057F}{t_1}\int_0^t e^{-30(t-\tau)}\sin 91.1039(t-\tau)\,d\tau$$

$$y(t) = \frac{0.057F}{t_1}\int_0^{t_1} e^{-30(t-\tau)}\sin 91.1039(t-\tau)\,d\tau \quad ;t_1 \le t \le t_2 \text{ and}$$

$$\frac{0.057F}{t_2 - t_1}\int_{t_1}^{t} e^{-30(t-\tau)}\sin 91.1039(t_2-\tau)\,d\tau$$

$$y(t) = \frac{0.057F}{t_1}\int_0^{t_1} e^{-30(t-\tau)}\sin 91.1039(t-\tau)\,d\tau \quad ;t \ge t_2$$

$$\frac{0.057F}{t_1^2}\int_{t_2-t_1}^{t_2} e^{-30(t-\tau)}\sin 91.1039(t_2-\tau)\,d\tau\,(t_2-t_1)$$

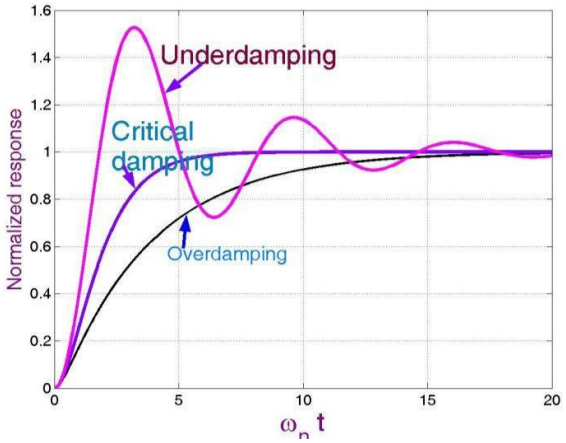
2.9.2. Second-order Time Response Specifications with step input

$$Y(s) = \frac{Kx}{m}\frac{1}{s(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})} \tag{2.85}$$

$$y(t) = \frac{Kx}{m\omega_n^2}\left[1 - e^{-\zeta\omega_n t}\cosh\omega_n\sqrt{\zeta^2-1}\,t - \frac{\zeta}{\sqrt{\zeta^2-1}}\sinh\omega_n\sqrt{\zeta^2-1}\,t\right] \tag{2.86}$$

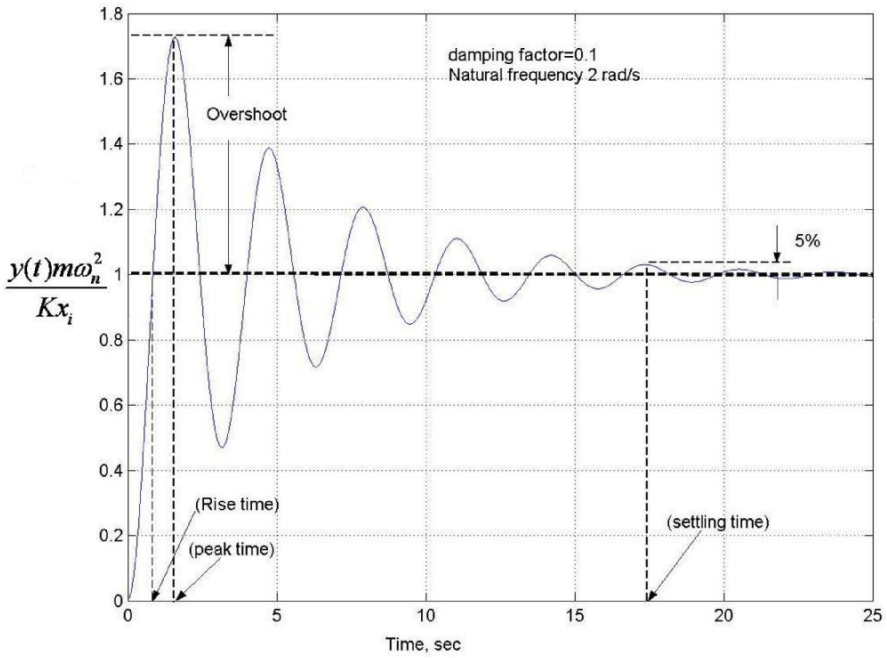
$$Y(s) = \frac{Kx}{m}\frac{1}{s(s+\zeta\omega_n-j\omega_d)(s+\zeta\omega_n+j\omega_d)} \tag{2.87}$$

$$y(t) = \frac{Kx}{m\omega_n^2}\left[1 - e^{-\zeta\omega_n t}\cos\omega_d t - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right] \tag{2.88}$$



2.3.6Time Response Specifications with step-input for under-damped case

For under-damped case, the step-response of a second-order is shown as follows



$$y(t) \quad \frac{Kx}{m_n^2} 1 \quad \frac{e^{nt}}{\sqrt{1 \quad ^2}} \sin(\quad _d \quad t \quad) \tag{2.89}$$

$$\tan^{1} \quad \frac{\sqrt{1 \quad ^2}}{} \tag{2.90}$$

For this case, different time-domain specifications are described below.

(i) Delay time, t_d

(ii) Rise time, t_r
(iii) Peak time, t_p
(iv) Peak overshoot, M_p
(v) Settling time

For unity step input,

(i)Delay time, t_d: It is the time required to reach 50% of output.

$$y(t_d) = \frac{1}{2} = 1 - \frac{e^{-\zeta w_n t_d}}{\sqrt{1-\zeta^2}}\sin(w_d t_d + \phi)$$

$$t_d = \frac{1 + 0.7\zeta}{w_n} \qquad (2.91)$$

(ii) Rise time, t_r: The time required by the system response to reach from 10% to 90% of the final value for over-damped case, from 0% to 100% of the final value for under-damped case and from 5% to 95% of the critically value for over-damped case.

.

$$y(t_r) = 1 = 1 - \frac{e^{-\zeta w_n t_r}}{\sqrt{1-\zeta^2}}\sin(w_d t_r + \phi)$$

$$\frac{e^{-\zeta w_n t_r}}{\sqrt{1-\zeta^2}}\sin(w_d t_r + \phi) = 0$$

$$w_d t_r = \pi$$

$$t_r = \frac{\pi}{w_d} \qquad (2.92)$$

(iii) Peak time, t_p: The time required by the system response to reach the first maximum value.

$$\frac{dy(t_p)}{dt} = 0$$

$$\frac{d\left(1 - \frac{e^{-\zeta w_n t_p}}{\sqrt{1-\zeta^2}}\sin(w_d t_p + \phi)\right)}{dt} = 0$$

$$\frac{d\left(\frac{e^{-\zeta w_n t_p}}{\sqrt{1-\zeta^2}}\sin(w_d t_p + \phi)\right)}{dt} = 0$$

$w_d t_p = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} + n\pi$; where $n = 1, 2, 3, \ldots$

For n=1,

$$w_d t_p = \pi$$

$$t_p = \frac{\pi}{w_d} \qquad (2.93)$$

(iv) Peak overshoot, M_p: It is the time required to reach 50% of output.

$$M_p\,\% = 100\,\frac{y(t_p) - 1}{1}$$

$$M_p \% \quad 100\ 1 \qquad \frac{e^{n\ tr}}{\sqrt{\quad 1^{\ 2}}}\sin(\ _d\ t_r \quad)\ 1$$

$$M_p \% \quad 100 \qquad \frac{e^{n\ tp}}{\sqrt{1^{\ 2}}}\sin(\ _d\ t_p \qquad)\ 100 \qquad \frac{e^{\ n\ \overline{\ _d}}}{\sqrt{\quad 1^{\ 2}}} \qquad \sin(\ _d\ t_p \quad)$$

$$M_p \% \quad 100 \qquad \frac{e^{\ 1\ \sqrt{\ }}}{\sqrt{\quad 1^{\ 2}}}\sin(\ _d \quad \sqrt{\quad 1^{\ 2}} \qquad)\ 100\frac{e^{\ 1\ \sqrt{\ }}}{\sqrt{1^{\ 2}}} \qquad \sin()$$

$$M_p \% \quad 100 \qquad \frac{e^{\ \sqrt{1^{\ 2}}}}{\sqrt{1^{\ 2}}} \qquad \sin 100\frac{e^{\ \sqrt{1^{\ 2}}}}{\sqrt{\quad 1^{\ 2}}}\sqrt{1^{\ 2}}$$

$$M_p \% \quad 100\ e^{\ \sqrt{\ }_{1^{\ 2}}} \tag{2.94}$$

(iv) Settling time, t_S: It is the time taken by the system response to settle down and stay with in 2% or 5% its final value.

For 2% error band,

$$t_s \quad \frac{4}{w_n} \tag{2.95}$$

For 5% error band,

$$t_s \quad \frac{3}{w_n} \tag{2.96}$$

Sl. No.	Time Specifications	
	Type	**Formula**
1	Delay time	$t_{\ d} \quad \frac{1\ 0.7}{w_n}$
2	Rise time	$t_r \quad \frac{\quad}{w_d}$
3	Peak time	$t_p \quad \frac{\quad}{w_d}$
4	Maximum overshoot	$M_p \% \quad 100\ e^{\ \sqrt{\ }_{1^{\ 2}}}$
5	Settling time	$t_s \quad \frac{4}{w_n}$

Solved Problems:

1. Consider the system shown in Figure 1. To improve the performance of the system a feedback is added to this system, which results in Figure 2. Determine the value of K so that the damping ratio of the new system is 0.4. Compare the overshoot, rise time, peak time and settling time and the nominal value of the systems shown in Figures 1 and 2.

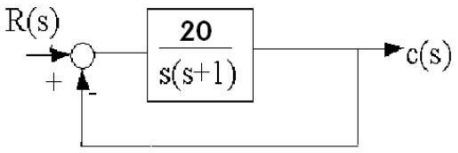


Figure 1

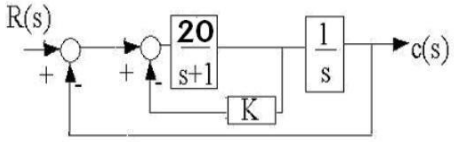


Figure 2

Solution:

For Figure 1,

$$\frac{c\ s}{R\ s} \quad \frac{G\ (s)}{1\ G\ (s)} \quad \frac{\frac{20}{s\ \ s\ \ 1}}{1\frac{20}{s\ s\ \ 1}} \quad \frac{20}{s^{2}\ \ s\ \ 20}$$

Here, $_{n}^{2}$ 20 and 2 $_{n}$ 1

$_{n}$ $\sqrt{20}$ rad/s and $\frac{1}{2\ _{n}} \quad \frac{1}{2\ \sqrt{20}}$ 0.112

For Figure 2,

$$\frac{c\ s}{R\ s} \quad \frac{G\ (s)}{1\ G\ (s)} \quad \frac{\frac{20}{s\ (s\ \ 1\ \ 20\ K\ )}}{1\frac{20}{s\ (s\ \ 1\ \ 20\ K\ )}} \quad \frac{20}{s^{2}\ \ 1\ \ 20\ K\ s\ \ 20}$$

Here, $_{n}^{2}$ 20 and 2 $_{n}$ 1 20K

$_{n}$ $\sqrt{20}$ rad/s

$$\frac{}{2\ _{n}} \quad \frac{}{2\sqrt{20}}$$

K 0.128

Transient characteristics of Figures 1 and 2

CharacteristicS	**Figure 1**	**Figure 2**
Overshoot, M_p	70%	25%
Rise time, t_r, sec	0.38	0.48
Peak time, t_p, sec	0.71	0.77
Settling time (2%), sec	8	2.24
Steady-state value, c_∞	1.0	1.0

1.1. Transient Response using MATLAB

Program 1: Find the step response for the following system $\frac{C(s)}{R(s)} = \frac{3s + 20}{s^2 + 5s + 36}$

Solution:

```
>> num=[3
20] num=
3         20
>> den=[1 5 36]
den=
1         5          36
>>sys=tf(num,den)
Transfer function:
      3s+20
--------------------
   s^2+5s+36
>>step(sys)
```

Program 2: Find the step response for the following system $\frac{C(s)}{R(s)} = \frac{20}{s^2 + 4s + 25}$

Solution:

```
>> num=[20]
num=
20
>> den=[1
425] den=
          1         425
>>sys=tf(num,den)
Transfer function:
20
--------------------
   s^2+4s+25
>>step(sys)
```

UNIT-3

3 Stability

3.1. Concept of stability

Stability is a very important characteristic of the transient performance of a system. Any working system is designed considering its stability. Therefore, all instruments are stable with in a boundary of parameter variations.

A linear time invariant (LTI) system is stable if the following two conditions are satisfied.

(i) **Notion-1:** When the system is excited by a bounded input, output is also bounded.

Proof:

A SISO system is given by

$$\frac{C(s)}{R(s)} = G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0} \tag{9.1}$$

So,

$$c(t) = g(t) * r(t) \tag{9.2}$$

Using convolution integral method

$$c(t) = \int_0^{\infty} g(\tau)\, r(t-\tau)\, d\tau \tag{9.3}$$

$g(t) = \mathcal{L}^{-1}[G(s)]$ = impulse response of the system

Taking absolute value in both sides,

$$|c(t)| = \left| \int_0^{\infty} g(\tau)\, r(t-\tau)\, d\tau \right| \tag{9.4}$$

Since, the absolute value of integral is not greater than the integral of absolute value of the integrand

$$\left|\int_0^{\infty} g(\tau)\, r(t-\tau)\, d\tau\right| \le \int_0^{\infty} |g(\tau)\, r(t-\tau)|\, d\tau$$

$$|c(t)| \le \int_0^{\infty} |g(\tau)|\, |r(t-\tau)|\, d\tau \tag{9.5}$$

$$|c(t)| \le \int_0^{\infty} |g(\tau)|\, |r(t-\tau)|\, d\tau$$

Let, r(t) and c(t) are bounded as follows.

$$\begin{aligned} |r(t)| &\le M_1 \\ |c(t)| &\le M_2 \end{aligned} \tag{9.6}$$

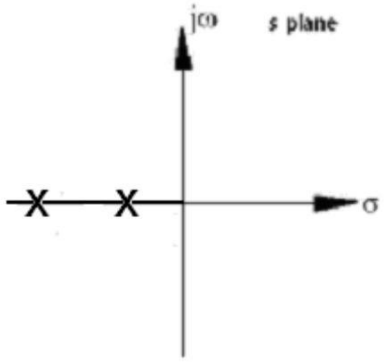
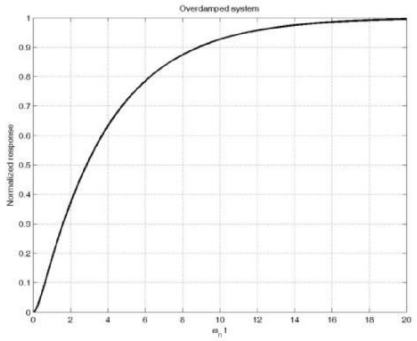
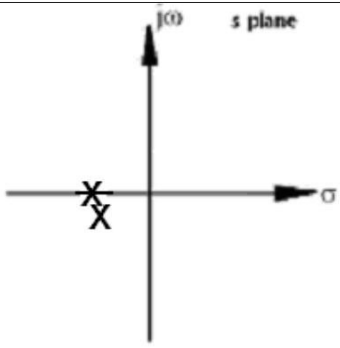
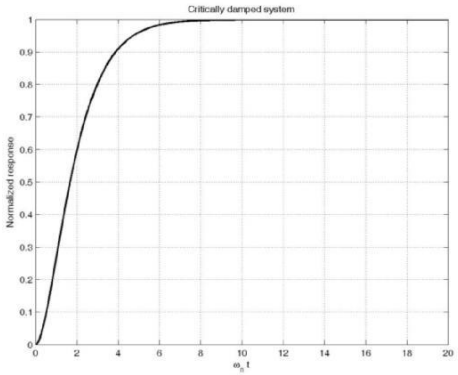
Then,

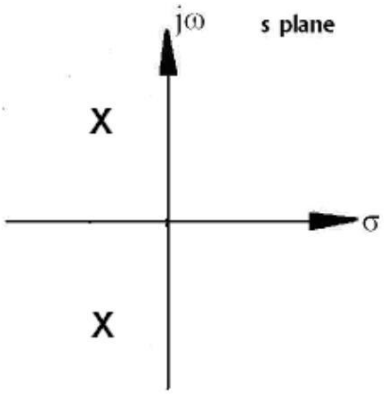
$$\left|c\ t\right| \quad \overset{M}{\underset{0}{}}_{1}\left|g \quad \right|d \quad M_{2} \tag{9.7}$$

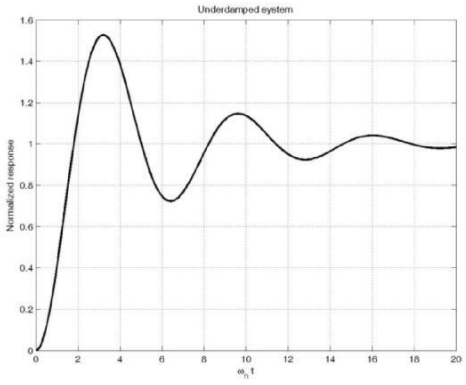
Hence, first notion of stability is satisfied if $\left|g \quad \right|d$ is finite or integrable.

(ii) **Notion-2:** In the absence of the input, the output tends towards zero irrespective of initial conditions. This type of stability is called asymptotic stability.

3.2. Effect of location of poles on stability

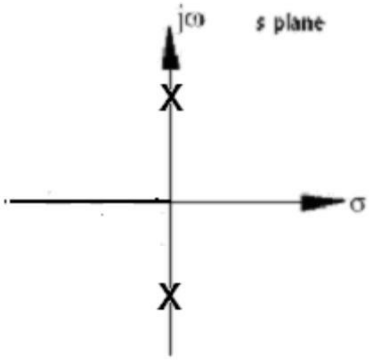
Pole-zero map	Normalized response
Over-damped close-loop poles	
	
Critically damped close-loop poles	
Pole-zero map	Normalized response
	
Under-dampedclose-loop poles	
Pole-zero map	Normalized response

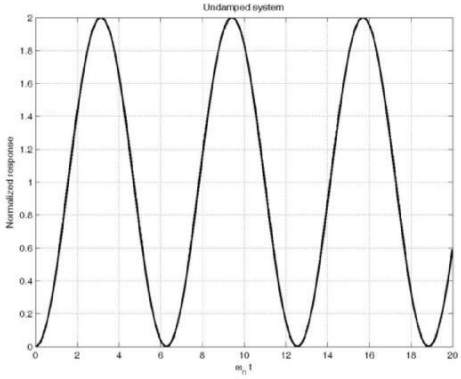




Un-dampedclose-loop poles

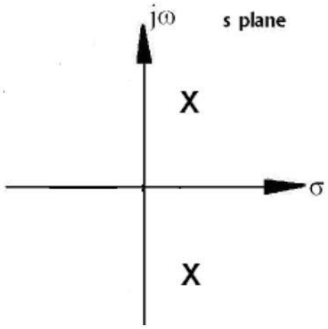
Pole-zero map	Normalized response

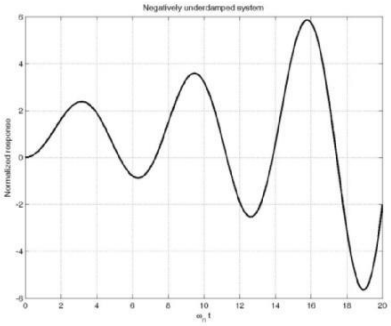




Negative Under-dampedclose-loop poles

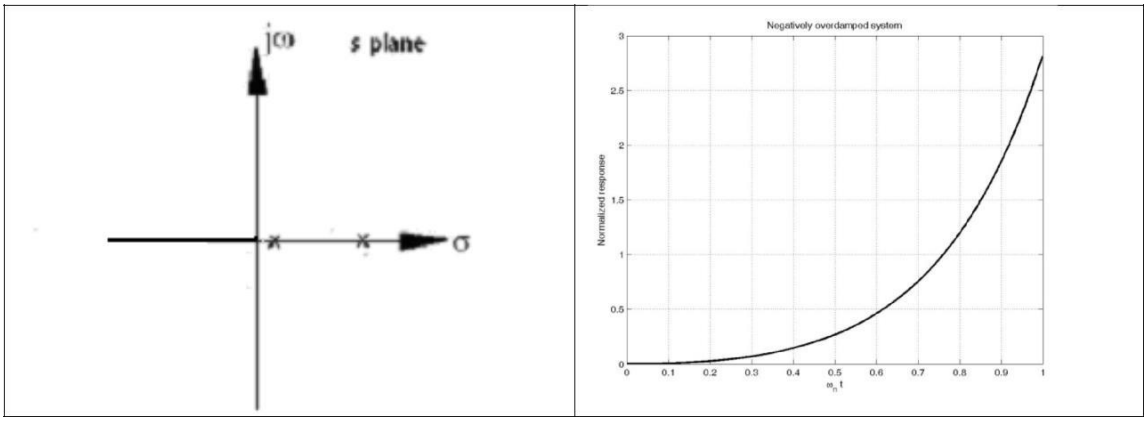
Pole-zero map	Normalized response





Negative Over-dampedclose-loop poles

Pole-zero map	Normalized response



3.3. Closed-loop poles on the imaginary axis

Closed-loop can be located by replace the denominator of the close-loop response with s=jω.

Example:

1. Determine the close-loop poles on the imaginary axis of a system given below.

$$G(s) \quad \frac{K}{s(s\ 1)}$$

Solution:

Characteristics equation, $B(s)\ s^2\ \ s\ \ K\ \ 0$

Replacing $s\ \ jw$

$B(j)\ \ (j)^2\ \ (j)\ \ K\ \ 0$

$(K\ \ ^2)\ j\ \ 0$

Comparing real and imaginary terms of L.H.S. with real and imaginary terms of R.H.S., we get

$\sqrt{K}$ and $\sqrt{0}$

Therefore, Closed-loop poles do not cross the imaginary axis.

2. Determinetheclose the imaginary axis of a system given below.

$B(s)\ s^3\ \ 6s^2\ \ 8s\ K\ 0$.

Solution:

Characteristics equation,

$B(j)\ \ (j)^3\ \ 6(j)^2\ \ 8j\ \ \ K\ \ 0$

$(K\ 6^2)\ \ \ \ j(8\ \ \ ^3)\ 0$

Comparing real and imaginary terms of L.H.S. with real and imaginary terms of R.H.S., we get

$\sqrt{8}$ rad/s and $\sqrt{K}$ $\qquad\qquad 6^2\ \ 48$

Therefore, Close-loop poles cross the imaginary axis for K>42.

3.4. Routh-Hurwitz's Stability Criterion

General form of characteristics equation,

$B(s) \quad a_n s^n a_{n-1} s^{n-1} \quad a_1 s \; a_0 \; 0$

$(s r_1)(s r_2) (s r_n) 0$

Where, r_i Roots of the characteristics equation

3.4.1. Necessary condition of stability:

Coefficients of the characteristic polynomial must be positive.

Example:

3. Consider a third order polynomial $B(s) s^3 3 s^2 16 s 130$. Although the coefficients of the above polynomial are positive, determine the roots and hence prove that the rule about coefficients being positive is only a necessary condition for the roots to be in the left s-plane.

Solution:

Characteristics equation, $B(s) s^3 3 s^2 16 s 130 0$ By

using Newton-Raphson's method $r_1 5$ and $r_{2,3} 1 j5$

Therefore, from the above example, the condition that coefficients of a polynomial should be positive for all its roots to be in the left s-plane is only a necessary condition.

3.4.2. Sufficient condition of stability:

.Method I (using determinants)

The coefficients of the characteristics equation are represented by determinant form as follows.

$$n \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix} \qquad (9.8)$$

Here, the determinant decreases by two along the row by one down the column. For stability, the following conditions must satisfy.

$$_1 \; a_{n-1} 0, \qquad _2 \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} 0, \qquad _3 \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix} 0 \qquad (9.9)$$

.2.Method II (using arrays)

The coefficients of the characteristics equation are represented by array form as follows.

$$\begin{array}{c|cc} s^n & a_n & a_{n-2} & a_{n-4} \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} \end{array} \qquad (9.10)$$

Where,

$$\begin{aligned} b_{n-1} &= \frac{(a_{n-1})(a_{n-2}) - a_n(a_{n-3})}{a_{n-1}} \\ b_{n-3} &= \frac{(a_{n-1})(a_{n-4}) - a_n(a_{n-5})}{a_{n-1}} \\ c_{n-1} &= \frac{(b_{n-1})(a_{n-3}) - a_{n-1}(b_{n-3})}{b_{n-1}} \end{aligned} \qquad (9.11)$$

For stability, the following conditions must satisfy.
The number of roots of B(s) with positive real parts is equal to the number of sign changes a_n, a_{n-1}, b_{n-1}, c_{n-1}, etc.

Example:

4. Find stability of the following system given by $G(s) = \frac{K}{s(s+1)}$ and $H(s) = 1$ using Routh-Hurwitz stability criterion.

Solution:

In the system, $T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{K}{s(s+1)}}{1 + \frac{K}{s(s+1)}} = \frac{K}{s^2 + s + K}$

Method-I,

Characteristics equation, $B(s) = s^2 + s + K = 0$

Here, $\begin{array}{c|cc} 1 & 1 & \\ 2 & 1 & K \end{array} \quad \begin{array}{c|c} 0 & \\ 0 & K \end{array}$

For stability, $\begin{array}{c} 1 \\ 2 \end{array} \begin{array}{c} 0 \\ 0 \end{array}$

The system is always stable for K>0.

Method-II,

Characteristics equation, $B\,s \quad s^2 \quad s \quad K \quad 0$

Here, Routh array is

$$\begin{array}{l|ll} s^2 & 1 & K \\ s^1 & 1 & 0 \\ s^0 & K & \end{array}$$

There are no sign changes in first column elements of this array.Therefore, the system is always stable for K>0.

1.10 Find stability of the following system given by $G(s) \quad \dfrac{K}{s(s\ 2)(s\ 4)}$ and $H(s)\ 1$ using Routh-Hurwitz stability criterion.

Solution:

In the system, $\dfrac{C(s)}{R\,s} \quad \dfrac{G\,s}{1\ G\,s\,H\,s} \quad \dfrac{\dfrac{K}{s(s\ 2)(s\ 4)}}{1\ \dfrac{K}{s(s\ \ 2)(s\ \ 4)}} \quad \dfrac{K}{s^3\ 6\,s^2\ 8s\ K}$

Method-I,

General form of characteristics equation, $B\,s\ a_3\,s^3\ a_2\,s^2\ a_1\,s\ a_0\ 0$ And in this system, characteristics equation is $B\,s\ s^3\ 6\,s^2\ 8\,s\,K\ 0$

Here,sufficient condition of stability suggests

$$_1\ \ 8\ \ 0,\ _2 \quad \begin{vmatrix} 6 & K \\ 1 & 8 \end{vmatrix} 48\ K\ \ 0,$$

$$_3 \quad \begin{vmatrix} 6 & K & 0 \\ 1 & 8 & 0 \\ 0 & 6 & K \end{vmatrix} K\,48\ K\ \ 0$$

Therefore, the system is always stable for $K\ \ 42$.

Method-II,

Characteristics equation is $B\,s \quad s^3\ \ 6\,s^2\ 8\,s\ \ K\ \ 0$

andRouth's array

$$\begin{array}{l|ll} s^3 & 1 & 8 \\ s^2 & 6 & K \\ s^1 & \dfrac{48\ K}{6} & 0 \\ s^0 & K & \end{array}$$

There are no sign changes in first column elements of this array if K 42. Therefore, the system is always stable for 0 K 42.

6. Find stability of the following system given by $B\,s\;s^3\;5\;s^2\;10\;s\;3$ using Routh-Hurwitz stability criterion.

Solution:

In this problem, given Characteristics equation is $B\,s \quad s^3 \;\; 5\,s^2 \;\; 10\,s \;\; 3 \quad 0$, andRouth's array is

$$\begin{array}{l|ll} s^3 & 1 & 10 \\ s^2 & 5 & 3 \\ s^1 & 9.4 & 0 \\ s^0 & 3 & \end{array}$$

There are no sign changes in first column elements of this array. Therefore, the system is always stable.

1.12. Find stability of the following system given by $B\,s\;s^3\;2s^2\;3s\;10$ using Routh-Hurwitz stability criterion.

Solution:

In this problem, given characteristics equation is

$B\,s \quad s^3 \;\; 2\,s^2 \;\; 3s \quad 10 \quad 0$ and

Routh's array is

$$\begin{array}{l|ll} s^3 & 1 & 3 \\ s^2 & 2 & 10 \\ s^1 & 2 & 0 \\ s^0 & 10 & \end{array}$$

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

8. Examine stability of the following system given by $s^5\;2s^4\;4s^3\;8s^2\;3s\;1$ using Routh-Hurwitz stability criterion.

Solution:

In this problem, Routh's array is

$$\begin{array}{l|lll} 5 & & & \\ s\;1 & & 4 & 3 \\ s^4 & 2 & 8 & 1 \\ s^2 & & & \\ s^1 & & & \\ s^0 & & & \end{array}$$

Here, the criterion fails. To remove the above difficulty, the following two methods can be used. Method-1

(i) Replace 0 by ε(very small number) and complete the array with ε.
(ii) Examine the sign change by taking0

Now, Routh's array becomes

s^5	1	4	3
s^4	2	8	1
s^3		2.5	0
s^2	5 8	1	0
s^1	5 8 / 2.5		
	5 8		
s^0	1		

Now putting 0 , Routh's array becomes

s^5	1	4	3
s^4	2	8	1
s^3		2.5	0
s^2	5 8	1	0
s^1	5 8 / 2.5		
	5 8		
s^0	1		

There are two sign changes in first column elements of this array. Therefore, the system is unstable. Method-2

Replace s by $\frac{1}{Z}$. The system characteristic equation $s^5 \ 2s^4 \ 4s^3 \ 8s^2 \ 3s \ 1 \ 0$ becomes

$\frac{1}{Z^5} \ \frac{2}{Z^4} \ \frac{4}{Z^3} \ \frac{8}{Z^2} \ \frac{3}{Z} \ 1 \ 0$

$Z^5 \ 3Z^4 \ 8Z^3 \ 4Z^2 \ 2Z \ 1 \ 0$

Now, Routh's array becomes

s^5	1	8	2
s^4	3	4	1
s^3	1.11.67	1.67	0
s^2	3.25	1	0
s^1	0.385	0	0
s^0	1	0	0

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

9. Examine stability of the following system given by $s^5 \ 2s^4 \ 2s^3 \ 4s^2 \ 4s \ 8$ using Routh-Hurwitz stability criterion.

Solution:

In this problem, Routh's array is

s^5	1	2	4
s^4	2	4	8
s^3			
s^2			
s^1			
s^0			

Here, the criterion fails. To remove the above difficulty, the following two methods can be used.

The auxillary equation is

$$A(s) = 2s^4 + 4s^2 + 8$$

$$\frac{dA(s)}{ds} = 8s^3 + 8s$$

Now, the array is rewritten as follows.

s^5	1	2	4
s^4	2	4	8
s^3			
s^2	2	8	0
s^1	24	0	
s^0	8		

There are two sign changes in first column elements of this array. Therefore, the system is unstable.

10. Examine stability of the following system given by $s^4 + 5s^3 + 2s^2 + 3s + 1 = 0$ using Routh-Hurwitz stability criterion. Find the number of roots in the right half of the s-plane.

Solution:

In this problem, Routh's array is

s^4	1	2	2
s^3	5	3	0
s^2	1.4	2	
s^1	4.14	0	
s^0	2		

There are two sign changes in first column elements of this array. Therefore, the system is unstable. There are two poles in the right half of the s-plane.

3.5. Advantages of Routh-Hurwitz stability

(i) Stability can be judged without solving the characteristic equation
(ii) Less calculation time
(iii) The number of roots in RHP can be found in case of unstable condition
(iv) Range of value of K for system stability can be calculated
(v) Intersection point with the jw-axis can be calculated
(vi) Frequency of oscillation at steady-state is calculated

3.5.1. Advantages of Routh-Hurwitz stability

(i) It is valid for only real coefficient of the characteristic equation
(ii) Unable to give exact locations of closed-loop poles
(iii) Does not suggest methods for stabilizing an unstable system
(iv) Applicable only to the linear system

3.6Root locus

3.6.1. Definition:

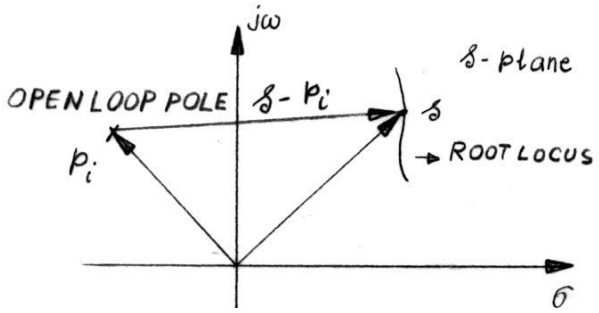
The locus of all the closed-loop poles for various values of the open-loop gain K is called **root locus**. The root-locus method is developed by W.R. Evans in 1954. It helps to visualize the various possibilities of transient response of stable systems. Closed-loop response function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{10.1}$$

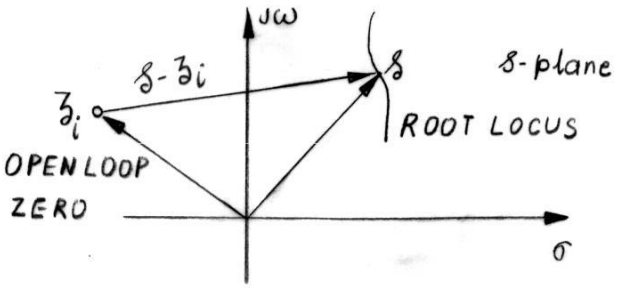
Characteristic equation

$$1 + G(s)H(s) = 1 + \frac{K(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)} = 0 \tag{10.2}$$

Vector from open-loop pole to the root-locus



Vector from open-loop zero to the root-locus



Behaviors of closed-loop poles

Closed-loop poles negative and real	Exponential decay	Stable
Closed-loop poles complex with negative real parts	Decaying and oscillatory	Stable
Closed-loop poles positive and real	Exponential increase	Unstable
Closed-loop poles complex with positive real parts	Exponential and oscillatory increase	Unstable

3.7. BASIS for CONSTRUCTION

3.7.1. Construction steps

1. Determine the number of open-loop poles and zeros
2. Mark open-loop poles and zeros on the s-plane
3. Determine parts of the root-locus on the real axis
4. Determine breakaway and break-in points

1.10 Draw asymptotes to the root-locus

1.11. Determine angles of departure

1.12. Determine angles of arrival

2. Determine points on the root-locus crossing imaginary axis
9. Obtain additional points and complete the root-locus

3.7.2. Starting points

Characteristics equation of a closed-loop system

$$1 \quad G(s)H(s) \quad 1 \quad \frac{K(s \quad z_1)(s \quad z_2)...(s \quad z_m)}{(s \quad p_1)(s \quad p_2)...(s \quad p_n)} \quad 0 \tag{10.3}$$

For K=0,

$$\frac{(s \quad p_1)(s \quad p_2)...(s \quad p_n) \quad K(s \quad z_1)(s \quad z_2)...(s \quad z_m)}{(s \quad p_1)(s \quad p_2)...(s \quad p_n)} \quad 0$$

$$(s \quad p_1)(s \quad p_2)...(s \quad p_n) \quad 0 \tag{10.4}$$

Open-loop poles are also closed-loop poles for K=0. A root-locus starts from every open-loop pole.

3.7.3. Ending points

Characteristics equation of a closed-loop system

$$1 \quad G(s)H(s) \quad 1 \quad \frac{K(s \quad z_1)(s \quad z_2)...(s \quad z_m)}{(s \quad p_1)(s \quad p_2)...(s \quad p_n)} \quad 0 \tag{10.5}$$

For K=∞,

$$1 \quad \frac{K(s \quad z_1)(s \quad z_2)...(s \quad z_m)}{(s \quad p_1)(s \quad p_2)...(s \quad p_n)}$$

$$(s \quad z_1)(s \quad z_2)...(s \quad z_m) \quad 0 \tag{10.6}$$

Root-locus ends at an open-loop zero or at infinity.

3.7.4. Magnitude and angle criterion

$$1 \quad G(s)H(s) \quad 1 \quad |G(s)H(s)|(\cos j \sin \quad) \quad 0 \tag{10.7}$$

Angle criterion:

$$\sum_{i\,1}^{n} \quad i \quad \sum_{j\,1}^{m} \quad j \quad 180^0 \quad 360k \tag{10.8}$$

Where, $_i$ angle in case of ith pole and $_j$ angle in case of jth zero

Magnitude criterion:

$$|G(s)H(s)| 1 \tag{10.9}$$

3.7.5 Determining gain at a root-locus point

Using the magnitude of vectors drawn from open-loop poles and zeros to the root-locus point, we get

$$\frac{\prod_{i=1}^{n}(s\ p_i)}{\prod_{j=1}^{m}(s\quad z_j)} \quad \frac{|(s\ p_1)\|(s\ p_2)|...|(s\ p_n)|}{|(s_{z_1})\|s\ z_2)|...|(s\ z_m)|}K \tag{10.10}$$

Gain at a root-locuspoint is determinedusing synthetic division.

Example:

Determine K of the characteristic equation for the root s=-0.81.10

Solution:

$$S^3\ \ 6s^2\ \ 8s\ \ K\ \ 0 \tag{10.11}$$

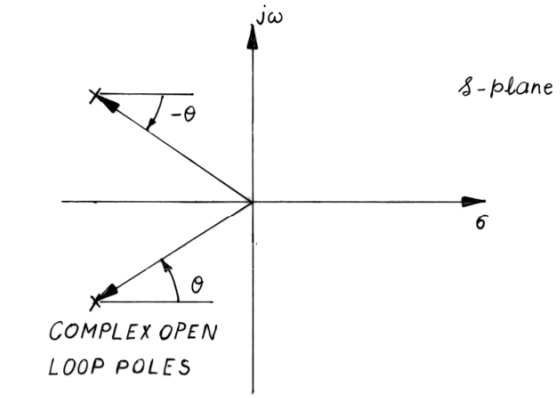
1	6	8	K
	-0.85	-1.878	-3.079
1	1.1015	3.622	K-3.079=0

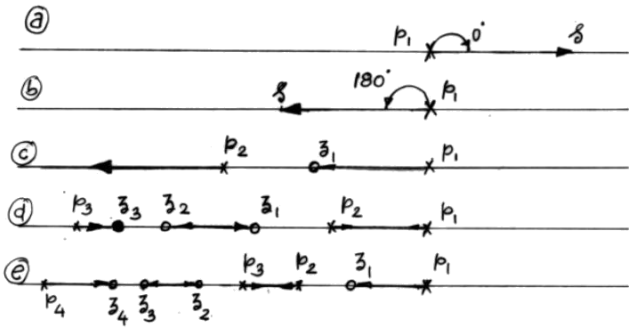
3.7.5 Determine parts of the root-locus on the real axis

1. Start from open-loop poles on the real axis, extend on the real axis for increasing values of the gain and end at an open-loop zero on the real axis.
2. Start from open-loop poles on the real axis, extend on the real axis for increasing values of the gain and end at an infinite value on the real axis.
3. Start from a pair of open-loop poles on the real axis, extend on the real axis for increasing values of gain, meet at a point and then leave the real axis and end at a complex open-loop zero or infinity.
4. Start from a pair of open-loop poles on the real axis, extend on the real axis for increasing values of gain, meet at a point and then leave the real axis. They may once again enter the real axis and end at open-loop zeros or at a large value on the real axis.
5. Start from a pair of complex open-loop poles, enter the real axis and end at an open-loop zero or an infinite value on the real axis. They could leave the real axis again and end at a complex open-loop zero or infinity.

3.8 Angle contributions from complex poles

Complex poles and zeros do not contribute to the angle criterion on the real axis





3.8.1. Determine breakaway and break-in points of the root-locus

$$1 + G(s)H(s) = 1 + \frac{KA(s)}{B(s)} = 0 \qquad (10.12)$$

$$f(s) = B(s) + KA(s) = 0 \qquad (10.13)$$

$$K = -\frac{B(s)}{A(s)} \qquad (10.14)$$

$$f(s) = (s - s_1)^r (s - s_2)\ldots(s - s_{n-r-1}) = 0$$

$$\frac{df(s)}{ds} = r(s - s_1)^{r-1}(s - s_2)\ldots(s - s_{n-r-1}) + (s - s_1)^r (s - s_3)\ldots(s - s_{n-r-1}) + \ldots$$

At breakaway and break-in points of the root-locus,

$$\left.\frac{df(s)}{ds}\right|_{s=s_1} = 0$$

$$f'(s) = B'(s) + KA'(s) = 0$$

$$K = -\frac{B'(s)}{A'(s)}$$

Therefore,

$$B'(s)A(s) - B(s)A'(s) = 0$$

(10.15)

(10.16)

(10.17)

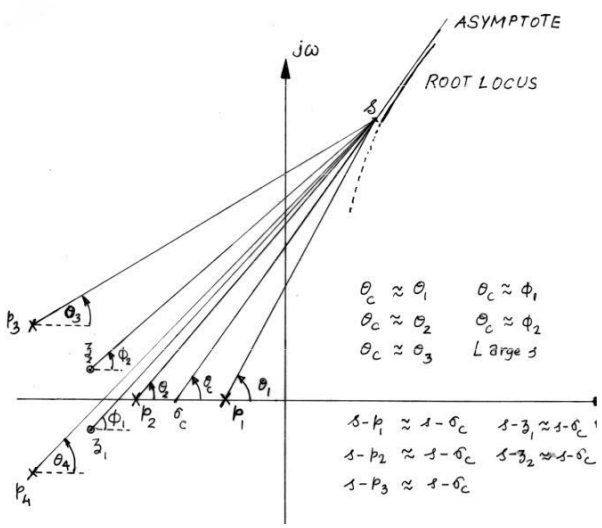
(10.18)

(10.19)

(10.20)

$$\frac{dK}{ds} \quad \frac{B'(s)A(s)}{A^2(s)} \quad B(s)A'(s) \quad 0 \tag{10.21}$$

3.8.2. Draw asymptotes to the root-locus



Angle of asymptotes

$$_c \quad \frac{180_0 \quad k360}{} \quad \text{where, } k=0, 1, 2, 3..$$

Location of asymptotes

$$K \frac{(s \ p_1)(s \ p_2)...(s \ p_n)(}{s \ z_1)(s \ z_2)...(s \ z_m)} \tag{10.22}$$

$$K \frac{s^n \ (p_1 \quad p_2 \quad ... \ p_n)s^{n-1} \ s^m}{(z_1 \quad z_2 \quad ...z_m)s^{m-1} \quad ...} \tag{10.23}$$

$$K \ s^{n-m} \ [(p \quad p \quad ... \ p \quad) \ (zz \quad ... \ z \quad)]s^{n-m-1} ... \tag{10.24}$$

$$s \ p_i \quad s \ _c \tag{10.25}$$

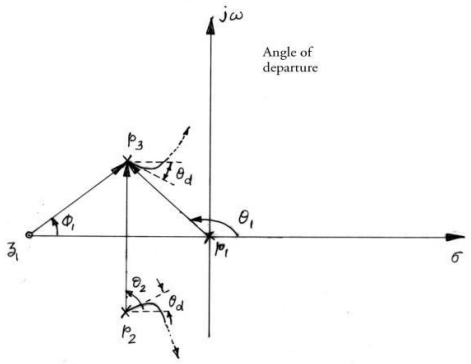
$$(s \ z_i) \ s \ _c \tag{10.26}$$

$$K \quad \frac{(s_c)^n}{(s \ _c)^m} s^{n-m} \quad (n \ m)_c s^{n-m-1} \quad ... \tag{10.27}$$

$$_c (p_1 p_2 ... p_n)(z_1 z_2 ... z_m)(n \quad m) \tag{10.28}$$

Angle of departure

$$_d \quad 180 \quad (1 \quad 2) \quad _1 \tag{10.29}$$

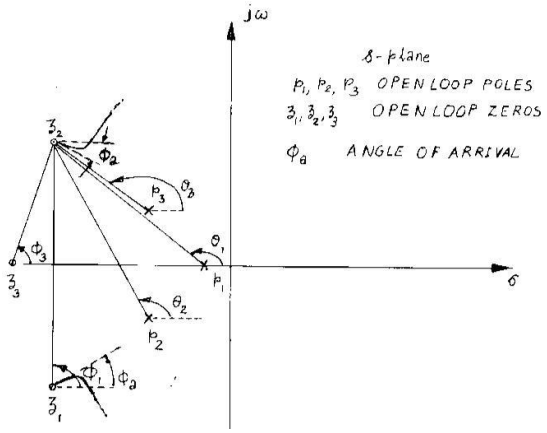


o

d=180 - angles of vectors to the complex open-loop pole in question from other open - loop poles + angles of vectors to the complex open-loop pole in question from all open-loop zeros

Angle of arrival

$$a \quad 180 \ (1 \quad 3) \ (1 \quad 2 \quad 3) \tag{10.30}$$



o

a=180 - angles of vectors to the complex open-loop zero in question from other open- loop zeros + angles of vectors to the complex open-loop zero in question from all open-loop poles

Determine points on the root-locus crossing imaginary axis

$$\text{Re}\,al\,[1 \;\; G(j\;)H(j\;)] \;\; 0 \tag{10.31}$$

$$imaginary\,[1 \;\; G(j\;)H(j\;)] \;\; 0 \tag{10.32}$$

Example

Problem-1: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s)\ \dfrac{K}{s(s\ 1)}$

Solution:

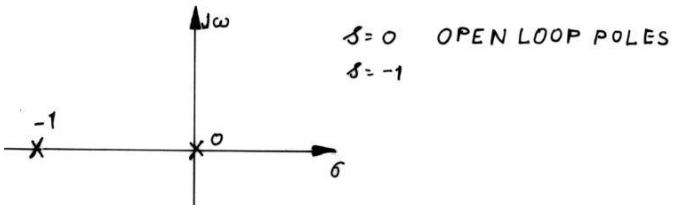
Step 1: Determine the number of open-loop poles and zeros

Number of open-loop poles n=2

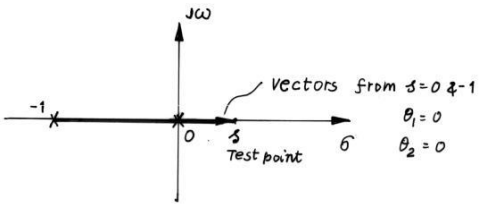
Number of open-loop zeros m=0

Open-loop poles: s=0 and s=-1

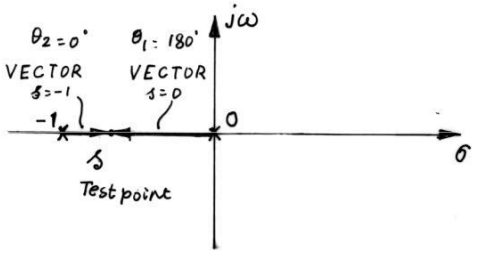
Step 2: Mark open-loop poles and zeros on the s-plane

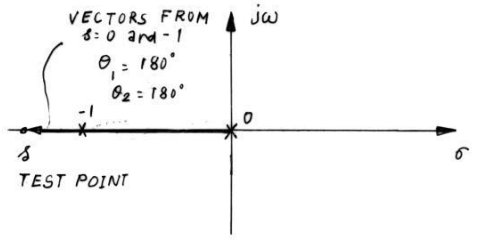


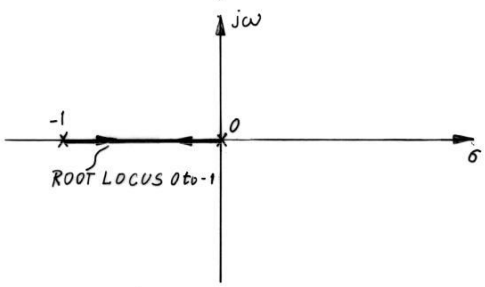
Step 3: Determine parts of the root-locus on the real axis Test points on the positive real axis



Test points in between the open-lop poles







Step 4: Determine breakaway and break-in point

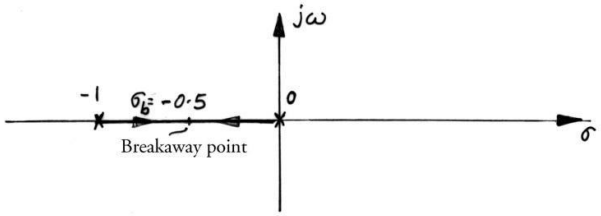
Characteristic equation, $K \quad s(s\ 1)$

$$\frac{dK}{ds} 2s \quad 1 \quad 0$$

breakaway point as σ_b =-0.5

Gain at the breakaway point

$K_b \mid 0.5 \quad 0 \parallel 0.5 \quad (1) \mid 0.25$



Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

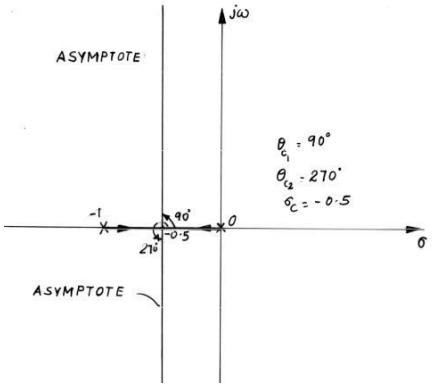
$$_c \quad _c \quad 180^0 \quad k\,360 \quad 180 \quad \frac{360k}{(n \quad m)2}$$

$$_c \quad 90^0 \quad k \quad 0$$

$$_c \quad 270^0 \quad k \quad 1$$

Centroid of asymptotes

$$_c \quad \frac{(p_1 \quad p_2 \quad \ldots p_n) \quad (z_1 \quad z_2 \quad \ldots z_m) \quad 0 \quad 1}{(n \quad m)2} \quad 0.5$$



Steps 6 & 7: Since there are no complex open-loop poles or zeros, angle of departure and arrival need not be computed

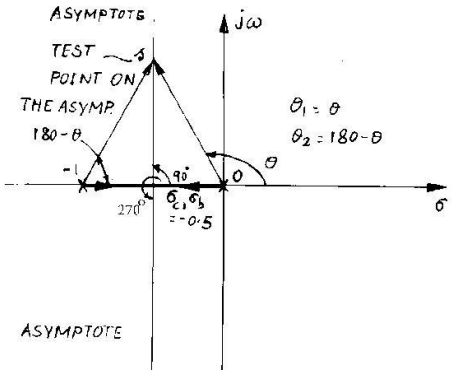
Step 8: Determine points on the root-locus crossing imaginary axis

$$1 \quad GH \quad 1 \quad \frac{K}{s^2 \quad s \quad K} \quad 0$$

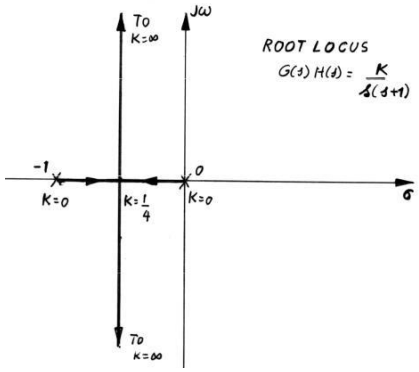
$$B(j \quad) \quad (j \quad)^2 \quad (j \quad) \quad K \quad (K \quad ^2) \quad j$$

$$K \quad ^2 \quad 0 \quad j \quad 0$$

The root-locus does not cross the imaginary axis for any value of K>0



Here, $s \frac{\sqrt{\quad}}{1 \sqrt{1\ 4K\ 2}}$

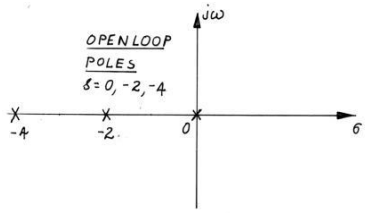


Problem-2: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s)\frac{K}{s(s\ 2)(s\ \ 4)}$

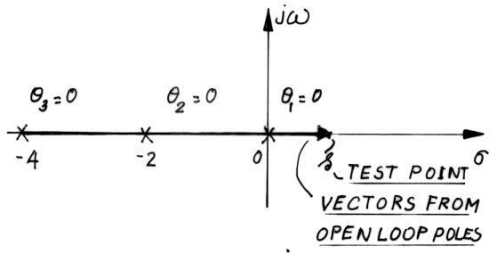
Solution:

Step 1: Determine the number of open-loop poles and zeros Number of open-loop poles n=3
Number of open-loop zeros m=0
Open-loop poles: s=0, s=-2 and s=-4

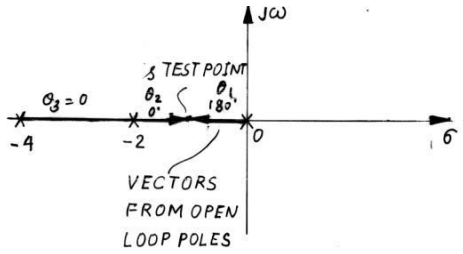
Step 2: Mark open-loop poles and zeros on the s-plane

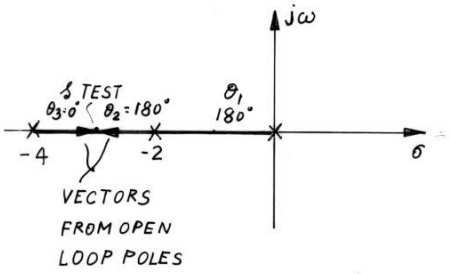


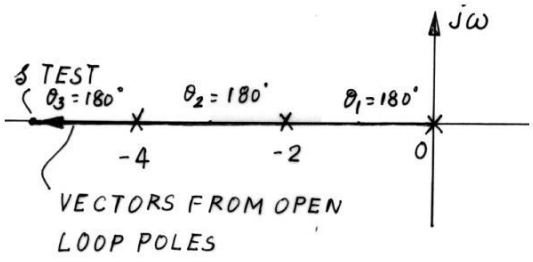
Step 3: Determine parts of the root-locus on the real axis Test points on the positive real axis

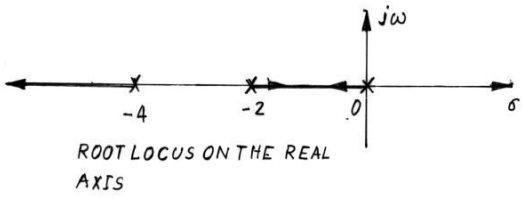


Test points in between the open-lop poles









Step 4: Determine breakaway and break-in point

Characteristic equation, $K \quad s(s \quad 2)(s \quad 4)$

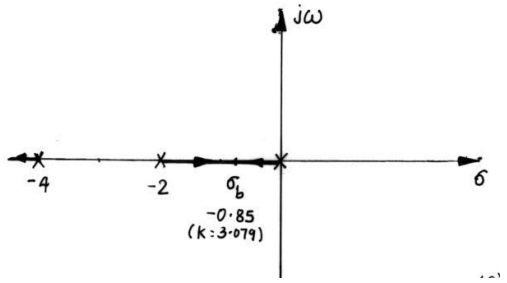
$\frac{dK}{} \quad (s \quad 2)(s \quad 4) \quad s(s \quad 4) \quad s(s \quad 2) \quad 0$

Breakaway point as σ_b=-0.85 and –3.15

σ_b = –3.15 is not on the root-locus and therefore not a breakaway or break-in point Gain at the breakaway point

$K_b \mid 0.85 \quad 0 \parallel 0.855 \quad (2) \parallel 0.85 \quad (4) \mid 3.079$

1	6	8	K
	-0.85	-1.878	-3.079
1	1.1015	3.622	K-3.079=0



Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

$$_c \quad 180^0 \quad k\,360 \quad 180 \quad 360k \quad m)3(n$$

$$_c \; 60^0 \; k \; 0 \; _c$$

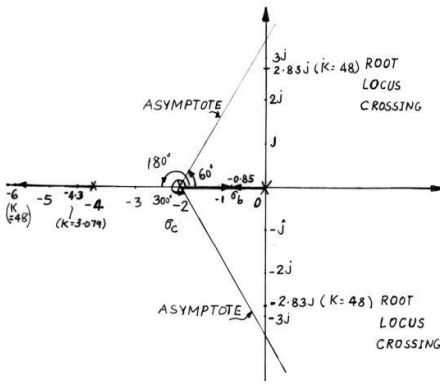
$$180^0 k\,1$$

$$_c \quad 300^0 \; k \; 2$$

Centroid of asymptotes

$$_c(p_1 \quad p_2 \quad \ldots p_n) \; (z_1 \quad z_2 \quad \ldots z_m) \; 0 \; 2 \; 4 \quad 2$$

$$(n \quad m)3$$



Steps 6 & 7: Since there are no complex open-loop poles or zeros, angle of departure and arrival need not be computed

Step 8: Determine points on the root-locus crossing imaginary axis

$$1 \; GH \; 1 \quad \frac{K}{s(s \; 2)(s \; 4)} s^3 \quad 6s^2 \quad 8s \quad K \; 0$$

$$B(j \;) \quad (j \;)^3 \quad 6(j \;)^2 \quad 8j \quad K \quad (K \quad 6^{\;2}) \quad j(8 \qquad ^3) \quad 0$$

When imaginary-part is zero, then $\sqrt{8}$ s $j\sqrt{8}$ and when real-part is zero, then $K \quad 6^{2} \quad 48$.
The root-locus does not cross the imaginary axis for any value of K>42.

1	6	8	48
	+j2.828	-8+j11.11.97	-48
1	6+j2.828	J11.11.97	0
1	6+j2.828	J11.11.97	
	-j2.828	-j11.11.97	
1	6	0	

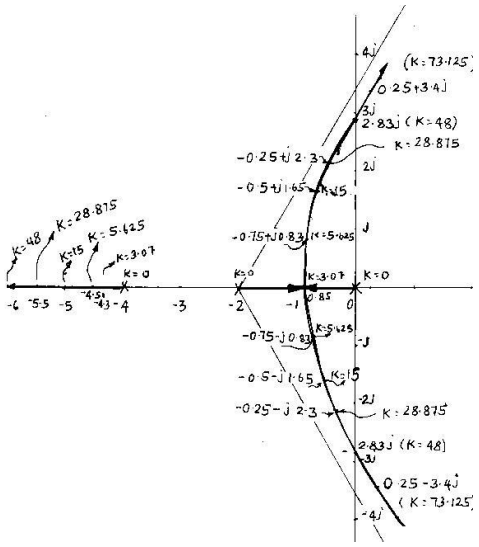
Therefore, closed-loop pole on the real axis for K=48 at $s \quad 6$

No.	Closed-loop pole on the real axis	K	Second and third closed- loop poles	Remarks
1	-1.809	3.07	-0.85,-0.85	Already computed
2	-4.50	1.1062 5	-0.75 j0.829	
3	- 1.1000	15	-0.5 j1.6583	
4	- 1.1050	22.875	-0.25 j2.2776	
5	- 1.11.00	48	j2.8284	Already computed
6	-1.11.5	73.125	0.25 j3.448	

Determine the gain corresponding to s=-4.5
K=|-4.5-(-4)||-4.5-(-2)||-4.5-0|=
1.10625
$s^{3} \quad 6s^{2} \quad 8s \quad K \quad 0$

1	6	8	K
	-4.5	-1.11.75	-1.10625
1	1.5	1.25	K- 1.10625=0

$(s^{2} \quad 1.5s \quad 1.25) \quad 0$
$s_{2,3} \quad 0.75 \quad j0.829$



Problem-3: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s)\frac{K}{s^2(s1)}$

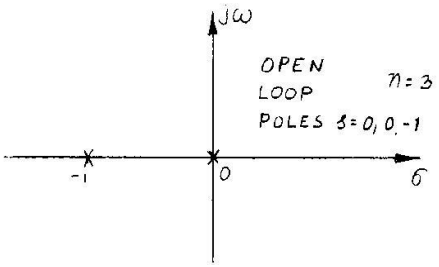
Solution:

Step 1: Determine the number of open-loop poles and zeros Number of open-loop poles n=3

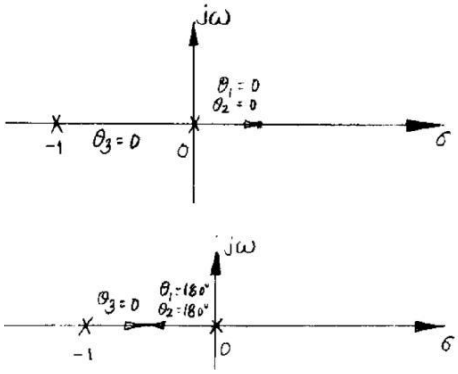
Number of open-loop zeros m=0

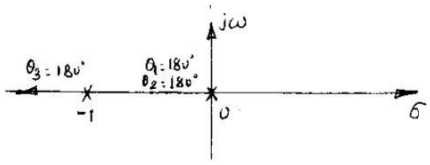
Open-loop poles: s=0, s=0 and s=-1

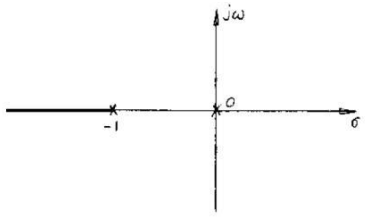
Step 2: Mark open-loop poles and zeros on the s-plane



Step 3: Determine parts of the root-locus on the real axis Test points on the positive real axis







Step 4: Determine breakaway and break-in point

Characteristic equation, $K \quad s^2 (s\ 1)$

$\frac{dK}{ds} 0$

$2 s (s\ 1) s\ 0\ s\ 2$

$s\ 3\ 0$

Breakaway point as σ_b= -2/3and 0

σ_b = -2/3is not on the root-locus and therefore not a breakaway or break-in point.

Therefore σ_b= 0 and the two loci start from the origin and breakaway at the origin itself.

Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

$_c \quad 180^0 \quad k\,360 \quad 180 \quad 360k$

$m)3(n$

$_c\ 60^0\ k\ 0\ _c$

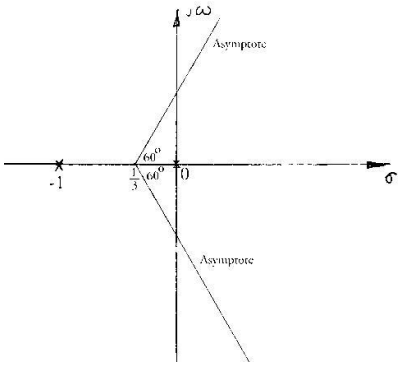
$180^0 k\ 1$

$_c \quad 300^0\ k\ 2$

Centroid of asymptotes

$_c (p_1 \quad p_2 \quad \ldots p_n) (z_1 \quad z_2 \quad \ldots z_m)\ 0\ 1\ 1 \quad _$

$(n\ m)33$



Steps 6 & 7: Since there are no complex open-loop poles or zeros, angle of departure and arrival need not be computed.

Step 8: Determine points on the root-locus crossing imaginary axis

$B(s)\ s^3\ \ s^2\ \ K$

$B(j\ )\ \ (j\ )^3\ \ (j\ )^2\ \ K\ \ (K\ \ ^2)\ \ j^3$

When imaginary-part is zero, then $0\ s\ 0$ and when real-part is zero, then K^2 0 .

The root-locus does not cross the imaginary axis for any value of K>0.

Additional closed-loop poles

No.	Closed-loop pole on the real axis	K	Second and third closed- loop poles
1	-1.5	1.125	0.25±j0.82
2	-2.0	4	0.50±j1.32
3	-2.5	9.375	0.75±j1.78
4	-3.0	18	1.00±j2.23

Determine the gain corresponding to s=-1.5

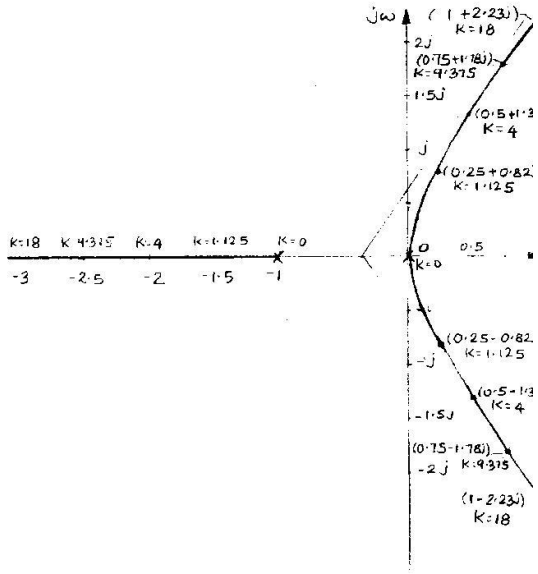
K=|-1.5-(-1)||-1.5-(0)||-1.5-0|= 1.125

$s^3\ \ s^2\ \ \ 1.125\ \ \ 0$

1	1	0	1.125
	-1.5	0.75	-1.125
1	-0.5	0.75	0

$(s^2\ 1.5\,s\ \ 1.25)\ \ \ 0$

$s_{2,3}\ \ \ 0.25\ \ \ j0.82$



Problem-4: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s)\frac{K}{s^4 \quad 5s^3 \quad 8s^2 \quad 6s}$

Solution:

Step 1: Determine the number of open-loop poles and zeros

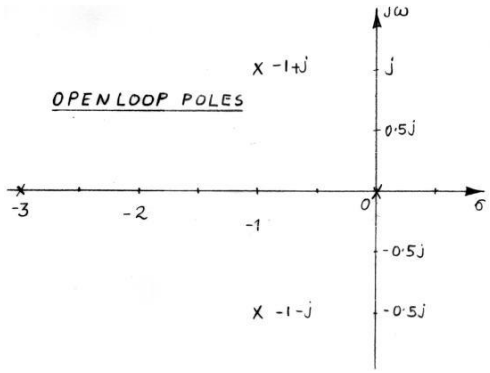
$s^4 \quad 5s^3 \quad 8s^2 \quad 6s \quad s(s^2 \quad 2s \quad 2)(s \quad 3) \quad (s \quad 1 \quad j)(s \quad 1 \quad j)(s \quad 3)s$

Number of open-loop poles n=4

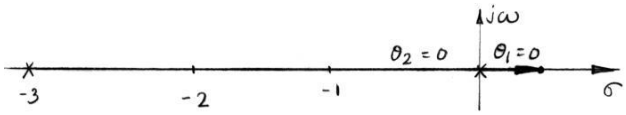
Number of open-loop zeros m=0

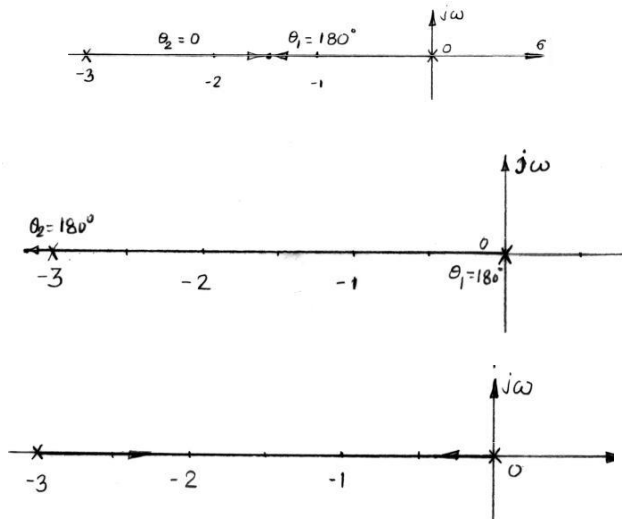
Open-loop poles: s=0 and s=-3, s=-1+j and s=-1-j

Step 2: Mark open-loop poles and zeros on the s-plane



Step 3: Determine parts of the root-locus on the real axis Test points on the positive real axis





Step 4: Determine breakaway and break-in point

Characteristic equation, $K \quad (s^4 \; 5s^3 \; 8s^2 \; 6s)$

$$\frac{dK}{ds} \quad 0$$

$$4s^3 \;\; 15s^2 \;\; 16s \;\; 6 \;\; 0$$

$$s^3 \; 3.75 s^2 \; 4s \; 1.5 \; 0 \quad f'(s)$$

$$3s^2 \; 1.12.5 s \; 4$$

This equation is solved using Newton-Raphson's method

$$s_{n\,1} \quad s_n \quad \frac{f(s_n)}{f'(s_n)}$$

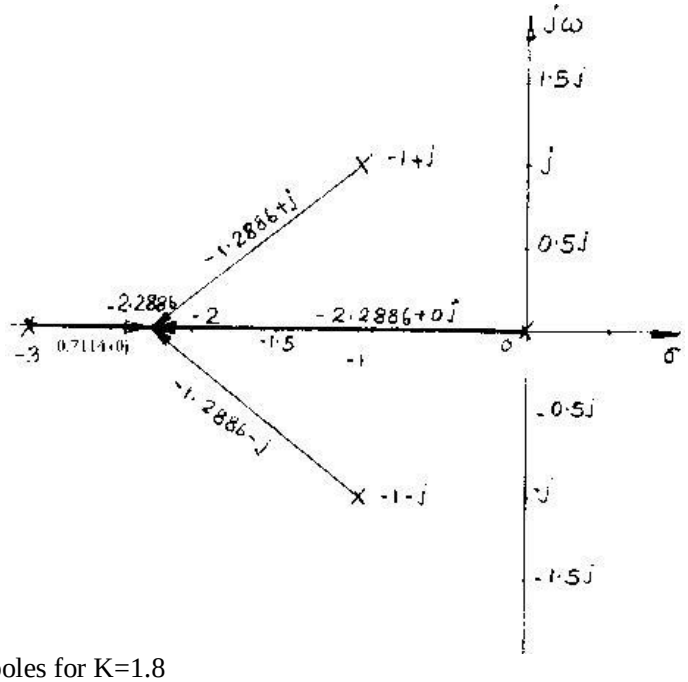
No.	s_n	$f(s_n)$	$f'(s_n)$	$s_{n\,1}$
1	-3.75	-13.5	12.0625	-3.0026
2	-3.0026	-3.7721	2.5273	-2.5602
3	-2.5602	-0.9421	4.4624	-2.3491
4	-2.3491	-0.1658	2.9364	-2.2926
5	-2.2926	-0.0103	2.5737	-2.2886
6	-2.2886	-1.1003×10^{-5}		

Breakaway point as σ_b= -2.3

Gain at the breakaway point, $K \;|\; 2.3 \quad (3) \| \; 2.3 \quad 0 \| \; 2.3 \quad (1 \quad j) \| \; 2.3 \quad (1 \quad j) \;|\; 1.83$

1	5	8	6	K
	-2.2886	-1.11.2053	-1.6073	-1.8316

1	2.7114	1.7947	1.8926	0



Other closed-loop poles for K=1.8

1	2.7114	1.7947	1.893
	-2.2886	-0.9676	-1.893
1	0.4228	0.8270	0

$s_{3,4}$ =-0.2114±j0.8814

Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

$$_c \quad \frac{180^0 \quad k\,360}{m)4(n} \quad \frac{180 \quad 360k}{}$$

$$_c \quad 45^0 \quad k \quad 0 \quad _c$$

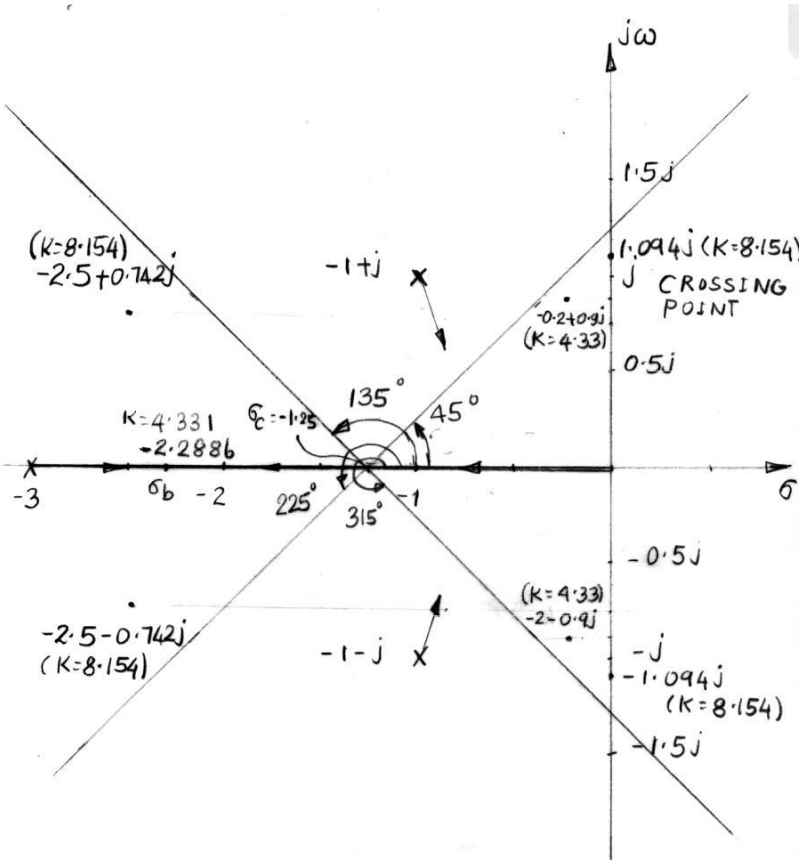
$$135 \quad ^0 \quad k \quad 1 \quad _c$$

$$225 \quad ^0 \quad k \quad 2 \quad _c$$

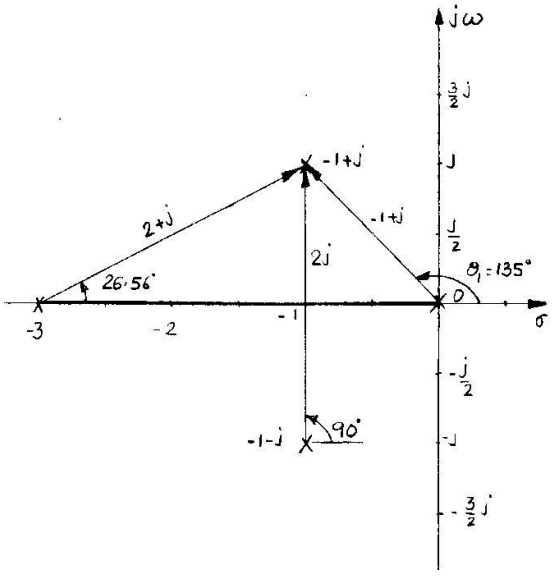
$$315^0 \, k \, 3$$

Centroid of asymptotes

$$_c \frac{(p_1 \quad p_2 \quad \ldots p_n) \; (z_1 \quad z_2 \quad \ldots z_m)}{(n \;\; m)4} \quad 0 \quad \frac{3 \;\; 1 \;\; j \;\; 1 \;\; j}{} \quad 1.25$$



Steps 6: Determine angles of departure



d 180^0 $(135^0$ $21.11.56^0$ $90^0)$ 71.56^0 282.44^0

Step 7: As there are no complex open-loop zeros, angle of arrival need not be computed.

Step 8: Determine points on the root-locus crossing imaginary axis

$B(s) = s^4 + 5s^3 + 8s^2 + 6s + K$

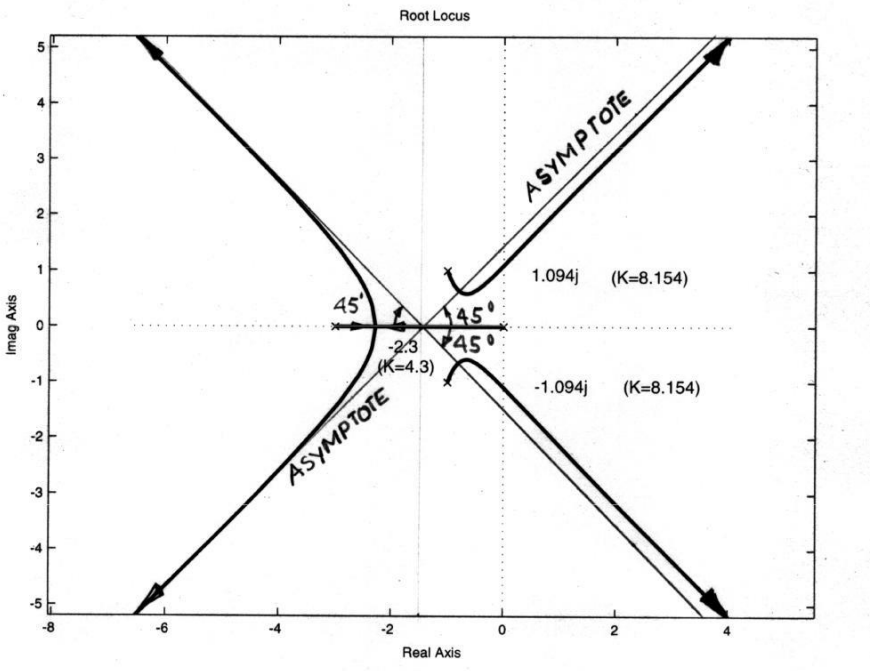
$B(j\omega) = (j\omega)^4 + 5(j\omega)^3 + 8(j\omega)^2 + 6j\omega + K = (\omega^4 - 8\omega^2 + K) + j(6\omega - 5\omega^3)$

When imaginary-part is zero, then $\omega = \sqrt{\frac{6}{5}}\, s \quad j\omega = \sqrt{\frac{6}{5}}$ and when real-part is zero,

then $K = 8\frac{6}{5} - \left(\frac{6}{5}\right)^2 = 2.16$.

There are two closed-loop poles on the imaginary axis for any value of K>0.
Additional closed-loop poles

No.	S_1	S_2	$S_{3,4}$	K
1	-0.25	-2.9217	-0.9142±0.7969j	1.0742
2	-0.50	-2.8804	-0.8098±0.655j	1.5625
3	-0.75	-2.8593	-0.6953±0.5938j	1.7930
4	-1.0	-2.8393	-0.5804±0.6063j	2.0000
5	-1.25	-2.8055	-0.4722±0.6631j	2.3242
6	-1.75	-2.6562	-0.3763±0.7354j	2.8125
7	-2.0	-2.5214	-0.2393±0.8579j	4.0



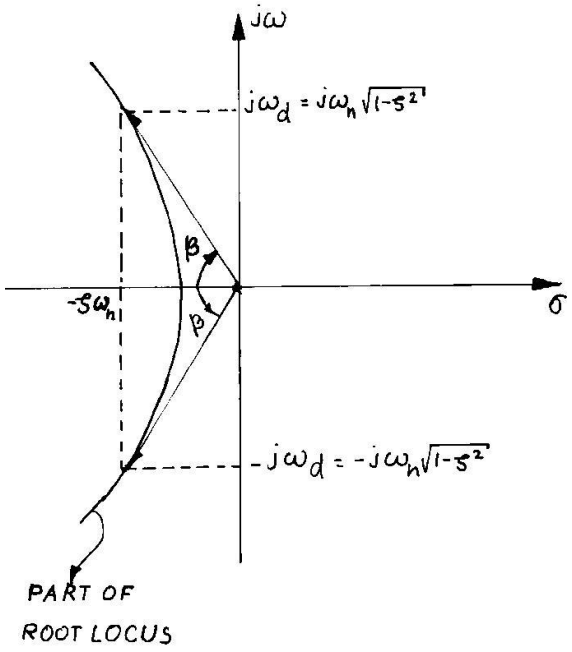
Additional Information from Root-Locus Plot

1. Gain Margin

$$GM \quad 20\log \frac{K_2}{K_1} \tag{10.33}$$

K_1 is the gain of a feedback system at some point on the root-locus
K_2 is the gain at which the system becomes unstable

2. Transient Characteristics



Where, $\tan \frac{\sqrt{1 \quad ^2}}{1}$

3. Percentage overshoot

$$M_p \quad e \quad / \tan \tag{10.34}$$

4. Settling time

$$t_s \frac{4}{n} \tag{10.35}$$

1.10 **Steady-state error** is also related to K.

Example

Problem-1: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s) \frac{K s^2 \quad 10 s \quad 100}{s^4 \quad 20 s^3 \quad 100 s^2 \quad 500 s \quad 1500}$, $H s \quad 1$

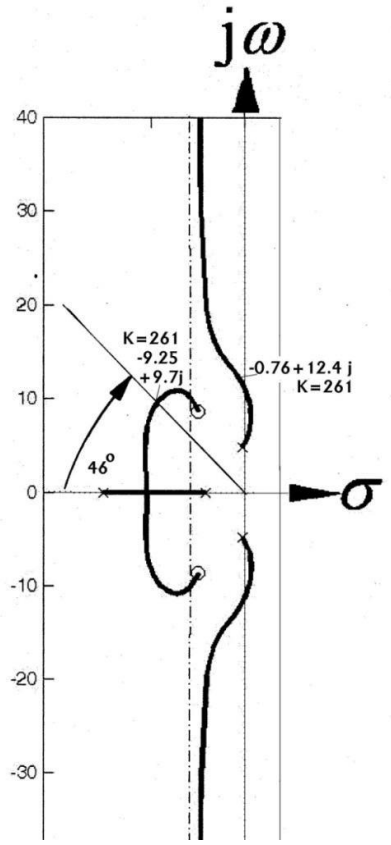
(a) Determine the value of gain at which the system will be stable and as well have a maximum overshoot of 5%.
(b) What is the gain margin at this point?
(c) What is the steady-state error for a unit step excitation at the above point?

Solution:

$$\tan \frac{\ln M_p}{} \quad 1.0487$$

(a) 46^0

$$\frac{1}{\sqrt{1 \quad \tan^2}} \quad 0.690 \qquad (10.36)$$



(b) $GM \; 20 \log \frac{192.2}{261} \; 2.65 dB$

(c) Position error

$$K_s \quad \lim_{s \; 0 \; s} \frac{K(s^2 \quad 10 s \quad 100)}{^4 \; 20 s^{\;3} \quad 100 s^{\;2} \; 500 s \quad 1500 \quad 1500} \quad \frac{100K}{}$$

Steady-state error,

$$S_{e}(\quad) \quad \frac{1}{1 \quad K_s} \quad \frac{1}{1 \quad 100K / 1500} \quad \frac{1500}{1500 \quad 100K}$$

$$S_e(\quad) \quad \frac{1500}{1500 \quad 100 \quad 261} \quad 1.104\%$$

Root locus

The locus of all the closed-loop poles for various values of the open-loop gain K is called **root locus**. The root-locus method is developed by W.R. Evans in 1954. It helps to visualize the various possibilities of transient response of stable systems.

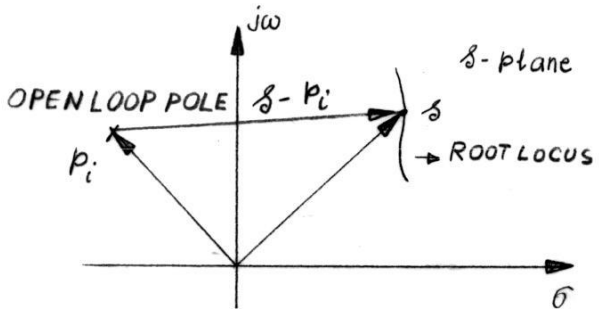
Closed-loop response function

$$\frac{C(s)}{R(s)} \quad \frac{G(s)}{1 \; G(s)H(s)} \tag{10.37}$$

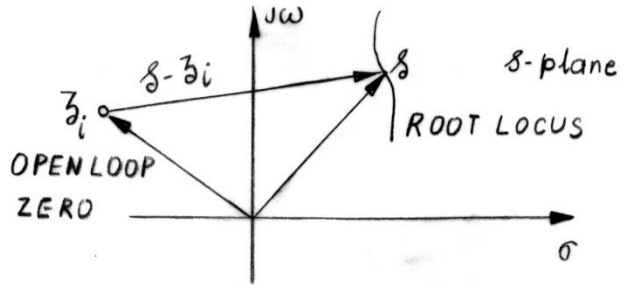
Characteristic equation

$$1 \; G(s)H(s) \; 1 \; \frac{K(s \; z_1)(s \; z_2)...(s \; z_m)}{(s \; p_1)(s \; p_2)...(s \; p_n)} \; 0 \tag{10.38}$$

Vector from open-loop pole to the root-locus



Vector from open-loop zero to the root-locus



Behaviors of closed-loop poles

Closed-loop poles negative and real	Exponential decay	Stable
Closed-loop poles complex with negative real parts	Decaying and oscillatory	Stable
Closed-loop poles positive and real	Exponential increase	Unstable
Closed-loop poles complex with positive real parts	Exponential and oscillatory increase	Unstable

BASIS for CONSTRUCTION

Construction steps

3.6Determine the number of open-loop poles and zeros

11. Mark open-loop poles and zeros on the s-plane
12. Determine parts of the root-locus on the real axis
13. Determine breakaway and break-in points
14. Draw asymptotes to the root-locus
15. Determine angles of departure
16. Determine angles of arrival
17. Determine points on the root-locus crossing imaginary axis
18. Obtain additional points and complete the root-locus

Starting points

Characteristics equation of a closed-loop system

$$1 + G(s)H(s) = 1 + \frac{K(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)} = 0 \tag{10.39}$$

For K=0,

$$(s - p_1)(s - p_2)\ldots(s - p_n) + K(s - z_1)(s - z_2)\ldots(s - z_m) = 0$$

$$(s - p_1)(s - p_2)\ldots(s - p_n) = 0 \tag{10.40}$$

Open-loop poles are also closed-loop poles for K=0. A root-locus starts from every open-loop pole.

Ending points

Characteristics equation of a closed-loop system

$$1 + G(s)H(s) = 1 + \frac{K(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)} = 0 \tag{10.41}$$

For K=∞,

$$1 + \frac{K(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)} = 0$$

$$(s - z_1)(s - z_2)\ldots(s - z_m) = 0 \tag{10.42}$$

Root-locus ends at an open-loop zero or at infinity.

Magnitude and angle criterion

$$1 + G(s)H(s) = 1 + |G(s)H(s)|(\cos\phi + j\sin\phi) = 0 \tag{10.43}$$

Angle criterion:

$$\sum_{i=1}^{n} \theta_i - \sum_{j=1}^{m} \phi_j = 180^0 + 360k \tag{10.44}$$

Where, θ_i angle in case of i^{th} pole and ϕ_j angle in case of j^{th} zero

Magnitude criterion: (10.45)

Determining gain at a root-locus point

Using the magnitude of vectors drawn from open-loop poles and zeros to the root-locus point, we get

$$\frac{\prod_{i=1}^{n}(s\ p_i)}{\prod_{j=1}^{m}(s\ z_j)} \quad \frac{|(s\ p_1)\|(s\ p_2)|...|(s\ p_n)|}{|(s\ z_1)\|s\ z_2)|...|(s\ z_m)|}K \tag{10.46}$$

Gain at a root-locus point is determined using synthetic division.

Example:
Determine K of the characteristic equation for the root s=-0.81.10

Solution:

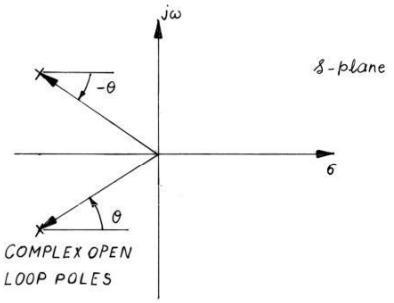
$$S^3\ \ 6s^2\ \ 8s\ \ K\ \ 0 \tag{10.47}$$

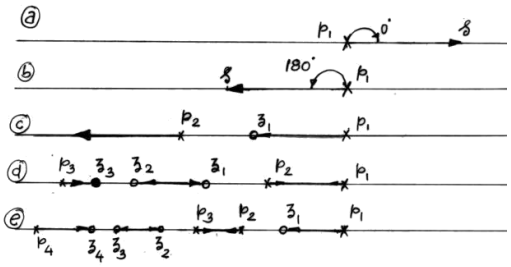
1	6	8	K
	-0.85	-1.878	-3.079
1	1.101 5	3.622	K-3.079=0

Determine parts of the root-locus on the real axis

6. Start from open-loop poles on the real axis, extend on the real axis for increasing values of the gain and end at an open-loop zero on the real axis.
7. Start from open-loop poles on the real axis, extend on the real axis for increasing values of the gain and end at an infinite value on the real axis.
8. Start from a pair of open-loop poles on the real axis, extend on the real axis for increasing values of gain, meet at a point and then leave the real axis and end at a complex open-loop zero or infinity.
9. Start from a pair of open-loop poles on the real axis, extend on the real axis for increasing values of gain, meet at a point and then leave the real axis. They may once again enter the real axis and end at open-loop zeros or at a large value on the real axis.
10. Start from a pair of complex open-loop poles, enter the real axis and end at an open-loop zero or an infinite value on the real axis. They could leave the real axis again and end at a complex open-loop zero or infinity.

Angle contributions from complex poles
Complex poles and zeros do not contribute to the angle criterion on the real axis





Determine breakaway and break-in points of the root-locus

$$1 + G(s)H(s) = 1 + \frac{A(s)}{} K = 0 \tag{10.48}$$

$$f(s) = B(s) + KA(s) = 0 \tag{10.49}$$

$$K = -\frac{B(s)}{A(s)} \tag{10.50}$$

$$f(s) = (s - s_1)^r (s - s_2)\ldots(s - s_{n-1}) = 0 \tag{10.51}$$

$$\frac{df(s)}{ds} = r(s - s_1)^{r-1}(s - s_2)\ldots(s - s_{n-1}) + (s - s_1)^r [(s - s_3)\ldots(s - s_{n-1}) + \ldots] \tag{10.52}$$

$$\frac{df(s)}{ds} \tag{10.53}$$

$$\left.\frac{df(s)}{ds}\right|_{s=s_1} = 0 \tag{10.54}$$

$$f'(s) = B'(s) + KA'(s) = 0 \tag{10.55}$$

$$K = -\frac{B'(s)}{A'(s)} \tag{10.56}$$

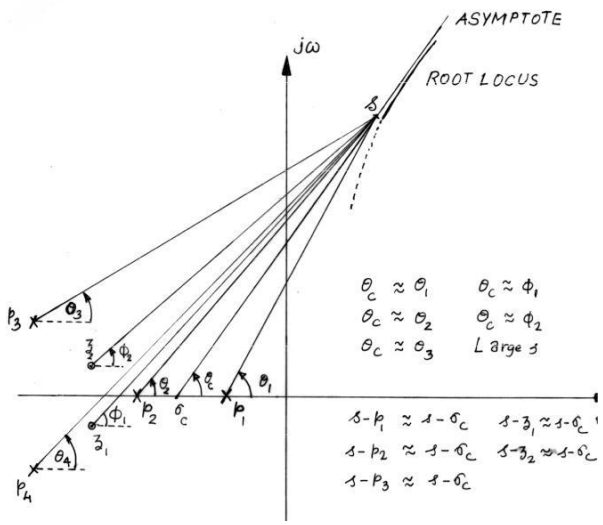
Therefore,

$$B'(s)A(s) - B(s)A'(s) = 0 \tag{10.57}$$

At breakaway and break-in points of the root-locus,

$$\frac{dK}{ds} = -\frac{B'(s)A(s) - B(s)A'(s)}{A^2(s)} = 0$$

Draw asymptotes to the root-locus



Angle of asymptotes

$$_c \frac{180_0 k360}{(n\ m)}$$ where, *k=0, 1, 2, 3..*

Location of asymptotes

$$K \frac{(s\ p_1)(s\ p_2)...(s\ p_n)(}{s\ z_1)(s\ z_2)...(s\ z_m)} \tag{10.58}$$

$$K \frac{s^n\ (p_1\ \ p_2\ \ ...\ p_n)s^{n\ 1}\ s^m}{(z_1\ \ z_2\ \ ...\ z_m)\ s^{m\ 1}\ \ ...} \tag{10.59}$$

$$K\ s^{n\ m}\ [(p\ \ p_{1\ \ 2}\ \ ...\ p_{n}\ )\ (zz_{1\ \ 2}\ \ ...\ z_{m}\ )]s^{n\ m\ 1}\ ... \tag{10.60}$$

$$s\ p_i\ \ s\ _c \tag{10.61}$$

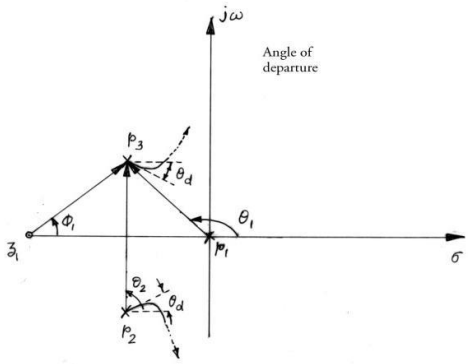
$$(s\ z_i)\ s\ _c \tag{10.62}$$

$$K\ \ \frac{(s_c)^n}{(s\ _c)^m} s^{n\ m}\ \ (n\ \ m)\ _c s^{n\ m\ 1}\ \ ... \tag{10.63}$$

$$_c (p_1 p_2 ... p_n)(z_1 z_2 ... z_m)(n \frac{}{m)} \tag{10.64}$$

Angle of departure

$$_d\ \ 180\ \ (_1\ \ \ _2)\ \ _1 \tag{10.65}$$

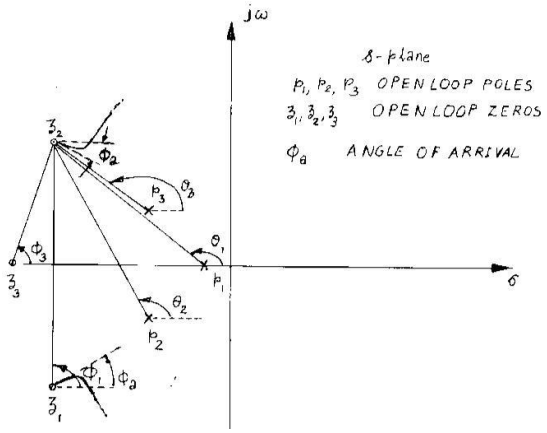


o

d=180 - angles of vectors to the complex open-loop pole in question from other open - loop poles + angles of vectors to the complex open-loop pole in question from all open-loop zeros

Angle of arrival

$$a \quad 180 \ (1 \quad 3) \ (1 \quad 2 \quad 3) \tag{10.66}$$



o

a=180 - angles of vectors to the complex open-loop zero in question from other open- loop zeros + angles of vectors to the complex open-loop zero in question from all open-loop poles

Determine points on the root-locus crossing imaginary axis

$$\text{Re}\,al\,[1 \;\; G(j\ )H(j\ )] \;\; 0 \tag{10.67}$$

$$imaginary\,[1 \;\; G(j\ )H(j\ )] \;\; 0 \tag{10.68}$$

Example

Problem-1: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s)\ \dfrac{K}{s(s\ 1)}$

Solution:

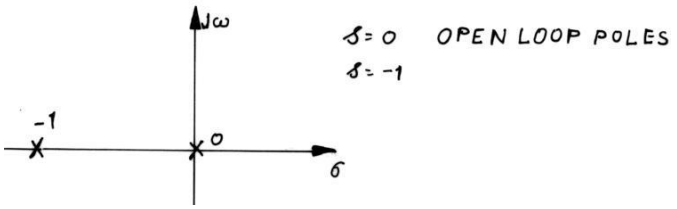
Step 1: Determine the number of open-loop poles and zeros

Number of open-loop poles n=2

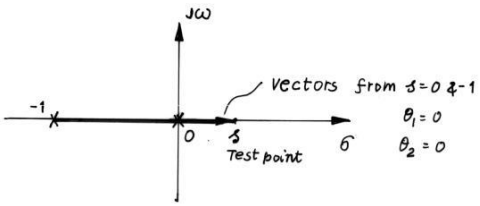
Number of open-loop zeros m=0

Open-loop poles: s=0 and s=-1

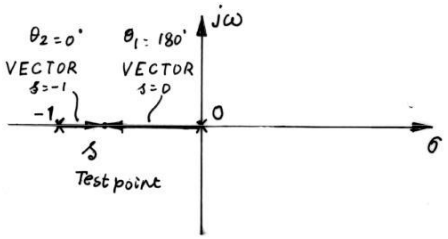
Step 2: Mark open-loop poles and zeros on the s-plane

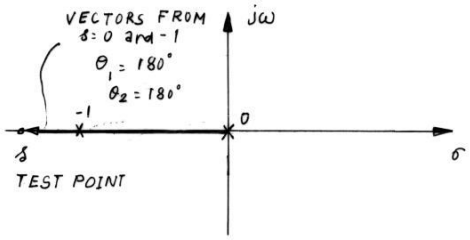


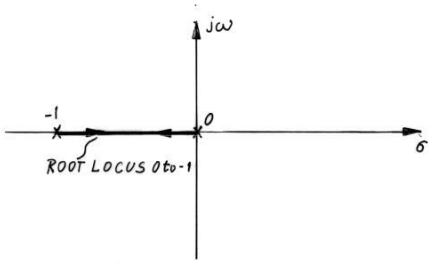
Step 3: Determine parts of the root-locus on the real axis Test points on the positive real axis



Test points in between the open-lop poles







Step 4: Determine breakaway and break-in point

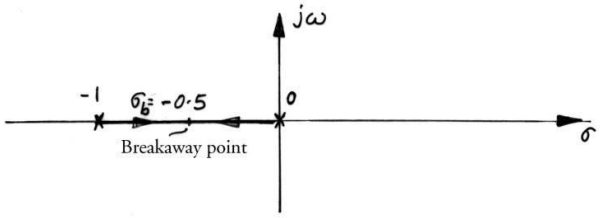
Characteristic equation, $K \quad s(s \quad 1)$

$\frac{dK}{ds} 2s \quad 1 \quad 0$

breakaway point as σ_b =-0.5

Gain at the breakaway point

$K_b \mid 0.5 \quad 0 \parallel 0.5 \quad (1) \mid 0.25$



Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

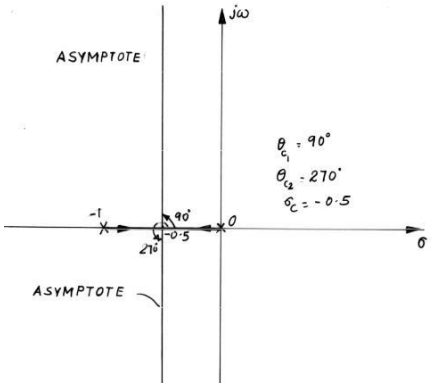
$$_c \quad _c \quad 180^0 \quad \frac{k\,360 \quad 180}{(n \quad m)2} \quad \frac{360k}{}$$

$$_c \quad 90^0 \quad k \quad 0$$

$$_c \quad 270^0 \quad k \quad 1$$

Centroid of asymptotes

$$_c \quad \frac{(p_1 \; p_2 \; \ldots p_n) \; (z_1 \; z_2 \; \ldots z_m) \; 0 \; 1}{(n \quad m)2} \quad \frac{}{0.5}$$



Steps 6 & 7: Since there are no complex open-loop poles or zeros, angle of departure and arrival need not be computed

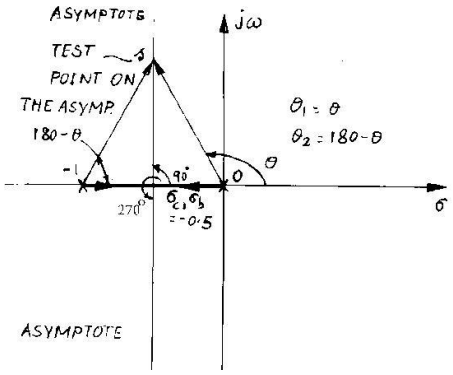
Step 8: Determine points on the root-locus crossing imaginary axis

$$1 \quad GH \quad 1 \quad \frac{K}{s(s\,1)} \quad s^2 \quad s \quad K \quad 0$$

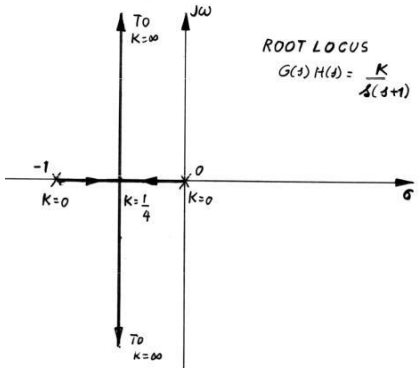
$$B(j \;) \quad (j \;)^2 \quad (j \;) \quad K \quad (K \quad ^2) \quad j$$

$$K \quad ^2 \quad 0 \quad j \quad 0$$

The root-locus does not cross the imaginary axis for any value of K>0



Here, $s \frac{\sqrt{}}{1 \sqrt{1\ 4K\ 2}}$



Problem-2: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s) \frac{K}{s(s\ 2)(s\ 4)}$

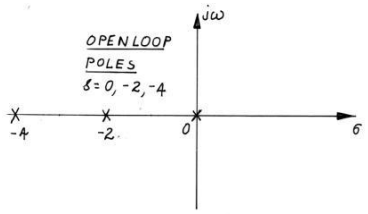
Solution:

Step 1: Determine the number of open-loop poles and zeros Number of open-loop poles n=3

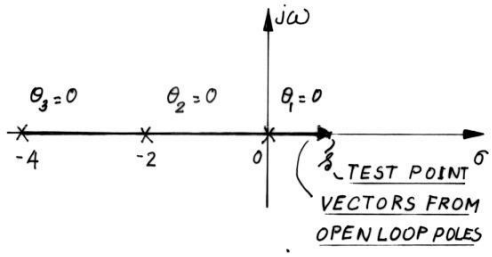
Number of open-loop zeros m=0

Open-loop poles: s=0, s=-2 and s=-4

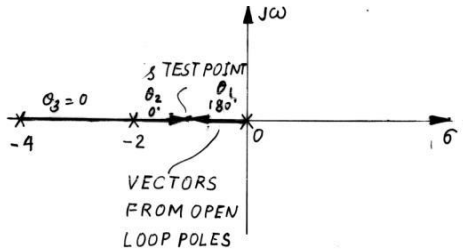
Step 2: Mark open-loop poles and zeros on the s-plane

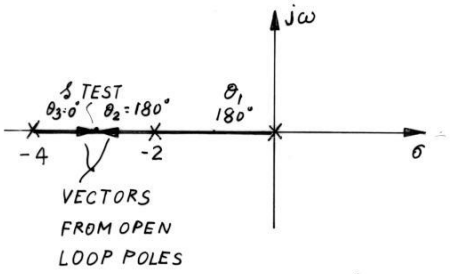


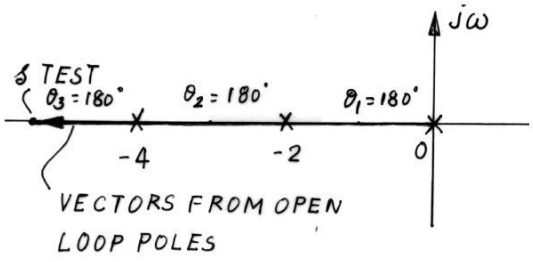
Step 3: Determine parts of the root-locus on the real axis Test points on the positive real axis

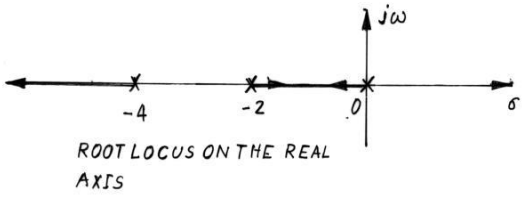


Test points in between the open-lop poles









Step 4: Determine breakaway and break-in point

Characteristic equation, $K \quad s(s \quad 2)(s \quad 4)$

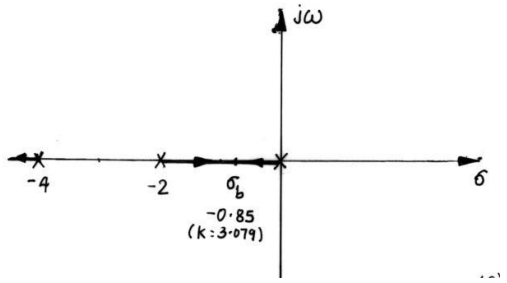
$\frac{dK}{} \quad (s \quad 2)(s \quad 4) \quad s(s \quad 4) \quad s(s \quad 2) \quad 0$

Breakaway point as σ_b=-0.85 and –3.15

σ_b = –3.15 is not on the root-locus and therefore not a breakaway or break-in point Gain at the breakaway point

$K_b \mid 0.85 \quad 0 \parallel 0.855 \quad (2) \parallel 0.85 \quad (4) \mid 3.079$

1	6	8	K
	-0.85	-1.878	-3.079
1	1.1015	3.622	K-3.079=0



Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

$$_c \quad \frac{180^0 \quad k\,360}{m)3(n} \quad \frac{180 \quad 360k}{}$$

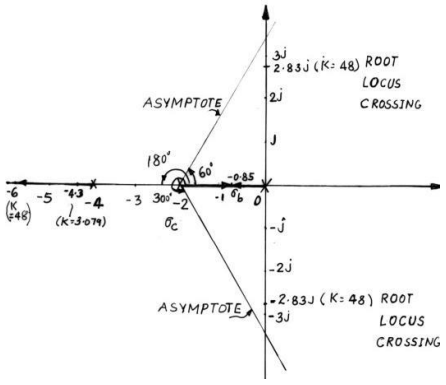
$$_c \; 60^{\;0} \; k \; 0 \; _c$$

$$180^{\;0} k \; 1$$

$$_c \quad 300^{\;0} \; k \; 2$$

Centroid of asymptotes

$$_c \frac{(p_1 \quad p_2 \quad \ldots p_n) - (z_1 \quad z_2 \quad \ldots z_m)}{(n \; m)3} \quad 0 \quad \frac{2 \; 4}{\quad 2}$$



Steps 6 & 7: Since there are no complex open-loop poles or zeros, angle of departure and arrival need not be computed

Step 8: Determine points on the root-locus crossing imaginary axis

$$1 \; GH \; 1 \quad \frac{K}{s\,(s \; 2)(s \; 4)} \; s^{\;3} \quad 6s^{\;2} \quad 8s \; K \; 0$$

$$B\,(j \;) \quad (j \;)^{\;3} \quad 6(j \;)^{\;2} \quad 8\,j \quad K \quad (K \quad 6^{\;2}) \quad j(8 \qquad ^{\;3}) \quad 0$$

When imaginary-part is zero, then $\sqrt{8}$ s $j\sqrt{8}$ and when real-part is zero, then K 6^2 48 .
The root-locus does not cross the imaginary axis for any value of K>42.

1	6	8	48
	+j2.828	-8+j11.11.97	-48
1	6+j2.828	J11.11.97	0
1	6+j2.828	J11.11.97	
	-j2.828	-j11.11.97	
1	6	0	

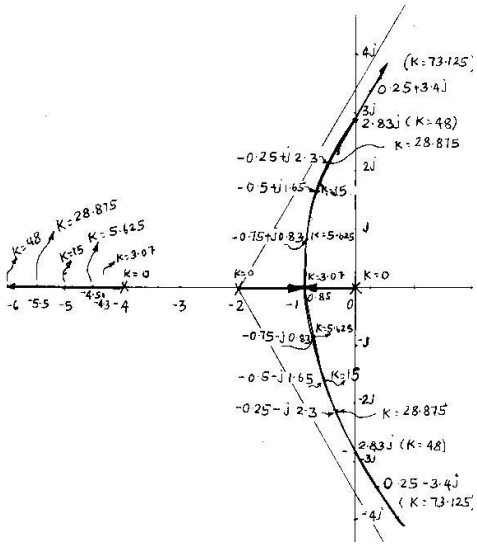
Therefore, closed-loop pole on the real axis for K=48 at s 6

No.	**Closed-loop pole on the real axis**	**K**	**Second and third closed- loop poles**	**Remarks**
1	-1.809	3.07	-0.85,-0.85	Already computed
2	-4.50	1.1062 5	-0.75 j0.829	
3	- 1.1000	15	-0.5 j1.6583	
4	- 1.1050	22.875	-0.25 j2.2776	
5	- 1.11.00	48	j2.8284	Already computed
6	-1.11.5	73.125	0.25 j3.448	

Determine the gain corresponding to s=-4.5
K=|-4.5-(-4)||-4.5-(-2)||-4.5-0|=
1.10625
s^3 6 s^2 8 s K 0

1	6	8	K
	-4.5	-1.11.75	-1.10625
1	1.5	1.25	K- 1.10625=0

(s^2 1.5 s 1.25) 0

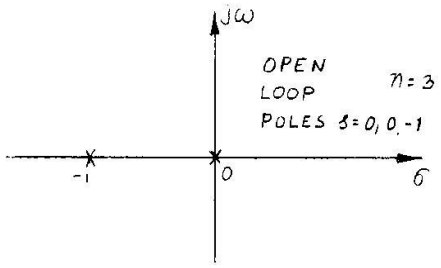


Problem-3: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s)\frac{K}{s^2(s1)}$

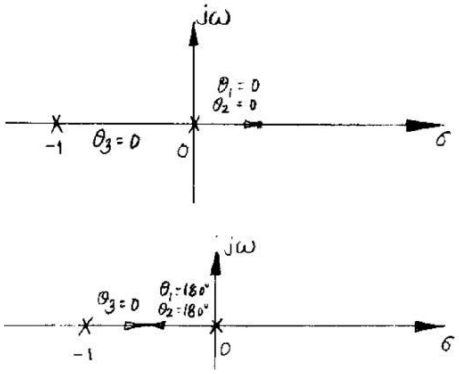
Solution:

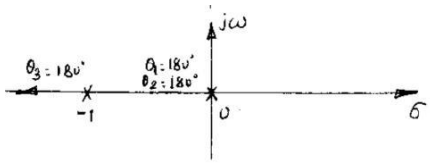
Step 1: Determine the number of open-loop poles and zeros Number of open-loop poles n=3
Number of open-loop zeros m=0
Open-loop poles: s=0, s=0 and s=-1

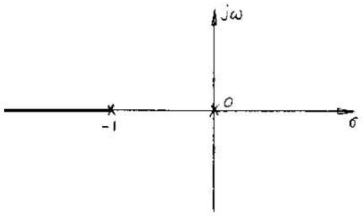
Step 2: Mark open-loop poles and zeros on the s-plane



Step 3: Determine parts of the root-locus on the real axis Test points on the positive real axis







Step 4: Determine breakaway and break-in point

Characteristic equation, $K \quad s^2 (s\ 1)$

$$\frac{dK}{ds} 0$$

$$2s(s\ 1)\ s\ 0\ s\ 2$$

$$s\ 3\ 0$$

Breakaway point as σ_b= -2/3and 0

σ_b = -2/3is not on the root-locus and therefore not a breakaway or break-in point.

Therefore σ_b = 0 and the two loci start from the origin and breakaway at the origin itself.

Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

$$_c \quad \frac{180^0 \quad k\,360}{m)3(n} \quad \frac{180 \quad 360k}{}$$

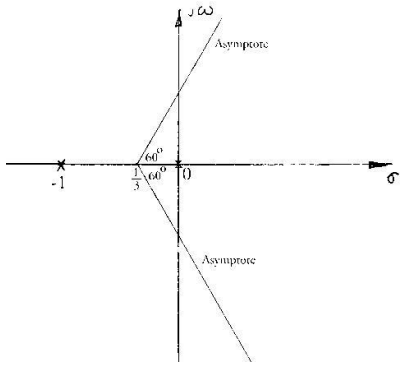
$$_c\ 60^{\ 0}\ k\ 0\ _c$$

$$180^{\ 0} k\ 1$$

$$_c \quad 300^{\ 0}\ k\ 2$$

Centroid of asymptotes

$$_c \frac{(p_1 \quad p_2 \quad \ldots p_n)\ (z_1 \quad z_2 \quad \ldots z_m)}{(n\ m)33}\ 0\ \frac{1\ \ 1}{}\ \ _$$



Steps 6 & 7: Since there are no complex open-loop poles or zeros, angle of departure and arrival need not be computed.

Step 8: Determine points on the root-locus crossing imaginary axis

$B(s)\ s^3\ \ s^2\ \ K$

$B(j\ )\ \ (j\ )^3\ \ (j\ )^2\ \ K\ \ (K\ \ ^2)\ \ j^3$

When imaginary-part is zero, then $0\ s\ 0$ and when real-part is zero, then K^2 0 .

The root-locus does not cross the imaginary axis for any value of K>0.

Additional closed-loop poles

No.	Closed-loop pole on the real axis	K	Second and third closed- loop poles
1	-1.5	1.125	0.25±j0.82
2	-2.0	4	0.50±j1.32
3	-2.5	9.375	0.75±j1.78
4	-3.0	18	1.00±j2.23

Determine the gain corresponding to s=-1.5

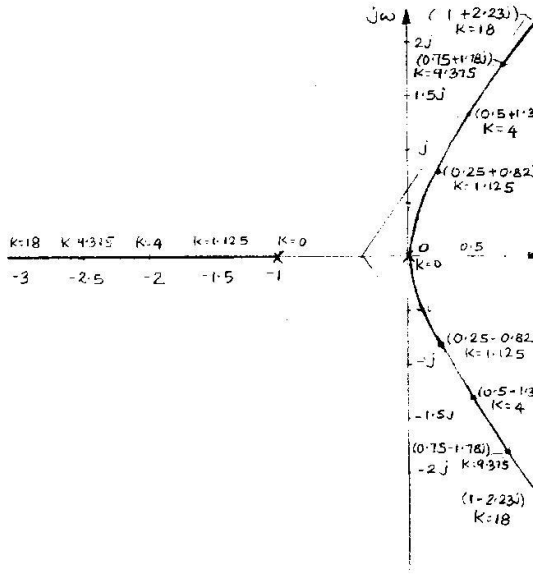
K=|-1.5-(-1)||-1.5-(0)||-1.5-0|= 1.125

$s^3\ \ s^2\ \ \ 1.125\ \ \ 0$

1	1	0	1.125
	-1.5	0.75	-1.125
1	-0.5	0.75	0

$(s^2\ 1.5\,s\ \ 1.25)\ \ \ 0$

$s_{2,3}\ \ \ 0.25\ \ \ j0.82$



Problem-4: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s)\frac{K}{s^4 \quad 5s^3 \quad 8s^2 \quad 6s}$

Solution:

Step 1: Determine the number of open-loop poles and zeros

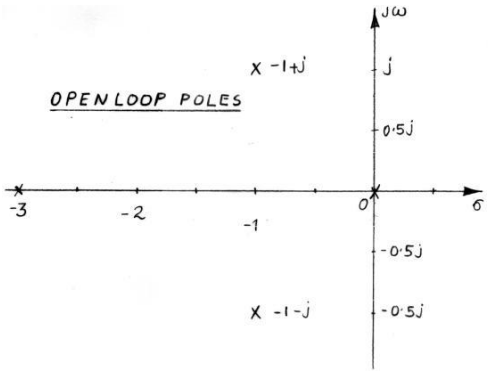
$s^4 \quad 5s^3 \quad 8s^2 \quad 6s \quad s(s^2 \quad 2s \quad 2)(s \quad 3) \quad (s \quad 1 \quad j)(s \quad 1 \quad j)(s \quad 3)s$

Number of open-loop poles n=4

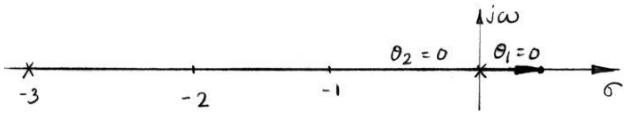
Number of open-loop zeros m=0

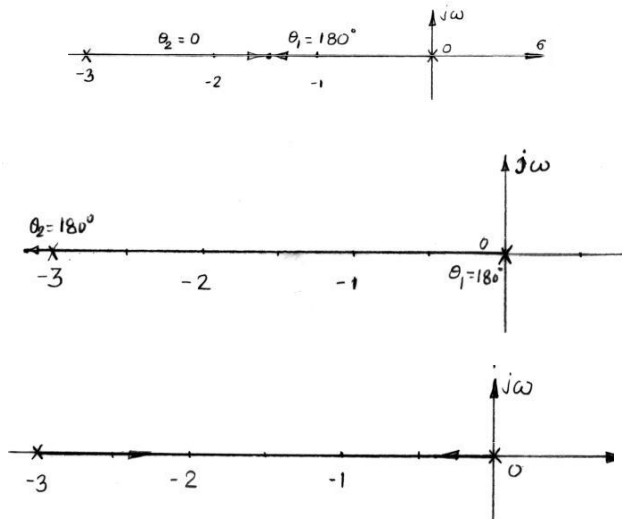
Open-loop poles: s=0 and s=-3, s=-1+j and s=-1-j

Step 2: Mark open-loop poles and zeros on the s-plane



Step 3: Determine parts of the root-locus on the real axis Test points on the positive real axis





Step 4: Determine breakaway and break-in point

Characteristic equation, $K \quad (s^4 \; 5s^3 \; 8s^2 \; 6s)$

$\frac{dK}{ds} \quad 0$

$4s^3 \;\; 15s^2 \;\; 16s \;\; 6 \;\; 0$

$s^3 \; 3.75 s^2 \; 4s \; 1.5 \; 0 \quad f'(s)$

$3s^2 \; 1.12.5 s \; 4$

This equation is solved using Newton-Raphson's method

$$s_{n\,1} \quad s_n \quad \frac{f(s_n)}{f'(s_n)}$$

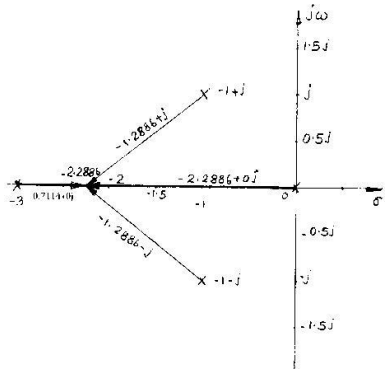
No.	s_n	$f(s_n)$	$f'(s_n)$	$s_{n\,1}$
1	-3.75	-13.5	12.0625	-3.0026
2	-3.0026	-3.7721	2.5273	-2.5602
3	-2.5602	-0.9421	4.4624	-2.3491
4	-2.3491	-0.1658	2.9364	-2.2926
5	-2.2926	-0.0103	2.5737	-2.2886
6	-2.2886	- 1.1003x 10^{-5}		

Breakaway point as σ_b= -2.3

Gain at the breakaway point, $K \; | \; 2.3 \quad (3) \; \| \; 2.3 \quad 0 \| \; 2.3 \quad (1 \quad j) \; \| \; 2.3 \quad (1 \quad j) \; | \; 1.83$

1	5	8	6	K
	-2.2886	-1.11.2053	-1.6073	-1.8316

1	2.7114	1.7947	1.8926	0



Other closed-loop poles for K=1.8

1	2.7114	1.7947	1.893
	-2.2886	-0.9676	-1.893
1	0.4228	0.8270	0

$s_{3,4}$ =-0.2114±j0.8814

Step 5: Draw asymptotes of the root-locus

Angle of asymptotes:

$$_c \quad \frac{180^0 \quad k\,360}{m)4(n} \quad \frac{180 \quad 360k}{}$$

$$_c \quad 45^0 \quad k \quad 0 \quad _c$$

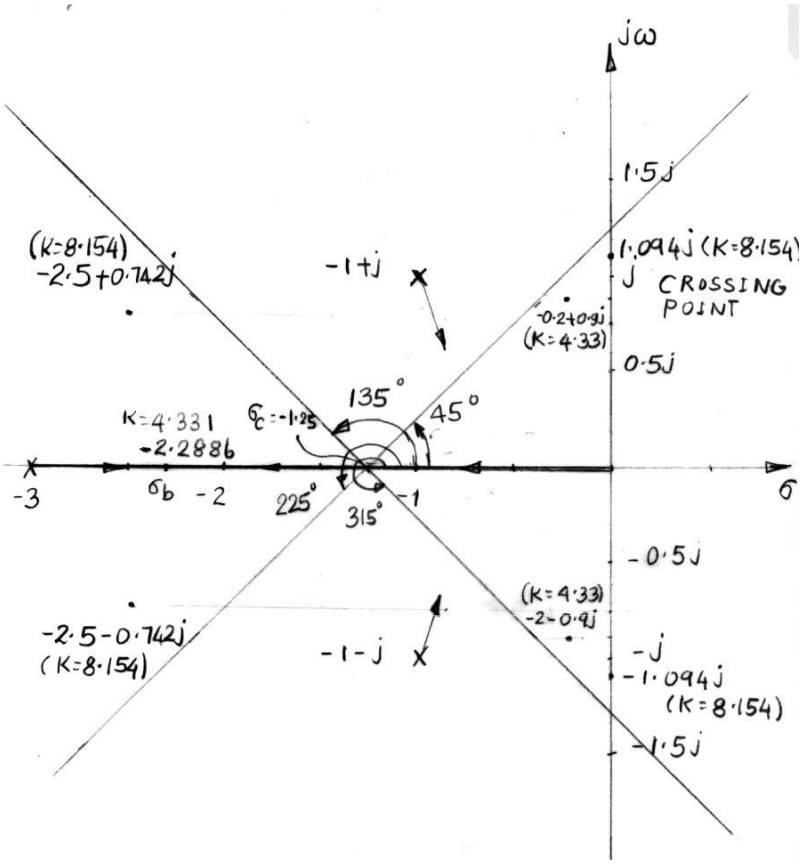
$$135 \quad ^0 \quad k \quad 1 \quad _c$$

$$225 \quad ^0 \quad k \quad 2 \quad _c$$

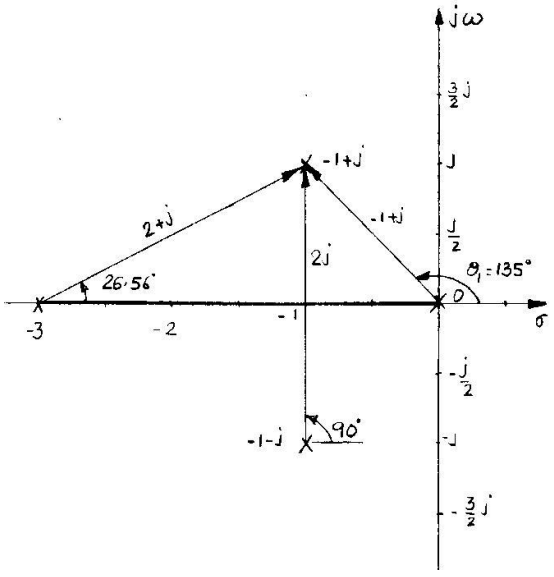
$$315^0 \, k \, 3$$

Centroid of asymptotes

$$_c \quad \frac{(p_1 \quad p_2 \quad \ldots p_n) \quad (z_1 \quad z_2 \quad \ldots z_m)}{(n \quad m)4} \quad 0 \quad \frac{3 \quad 1 \quad j \quad 1 \quad j}{} \quad 1.25$$



Steps 6: Determine angles of departure



d 180^0 $(135^0$ $21.11.56^0$ $90^0)$ 71.56^0 282.44^0

Step 7: As there are no complex open-loop zeros, angle of arrival need not be computed.

Step 8: Determine points on the root-locus crossing imaginary axis

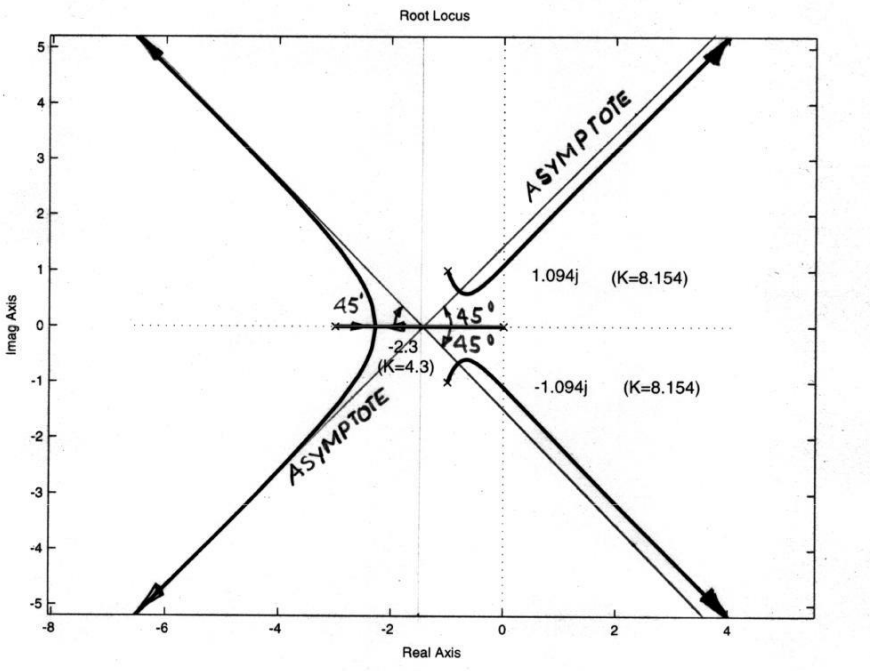
$B(s) = s^4 + 5s^3 + 8s^2 + 6s + K$

$B(j\omega) = (j\omega)^4 + 5(j\omega)^3 + 8(j\omega)^2 + 6j\omega + K = (\omega^4 - 8\omega^2 + K) + j(6\omega - 5\omega^3)$

When imaginary-part is zero, then $\omega = \pm\sqrt{\frac{6}{5}}$ and when real-part is zero, then $K = 8\frac{6}{5} - \left(\frac{6}{5}\right)^2 = 2.16$.

There are two closed-loop poles on the imaginary axis for any value of K>0.
Additional closed-loop poles

No.	S_1	S_2	$S_{3,4}$	K
1	-0.25	-2.9217	-0.9142±0.7969j	1.0742
2	-0.50	-2.8804	-0.8098±0.655j	1.5625
3	-0.75	-2.8593	-0.6953±0.5938j	1.7930
4	-1.0	-2.8393	-0.5804±0.6063j	2.0000
5	-1.25	-2.8055	-0.4722±0.6631j	2.3242
6	-1.75	-2.6562	-0.3763±0.7354j	2.8125
7	-2.0	-2.5214	-0.2393±0.8579j	4.0



Additional Information from Root-Locus Plot

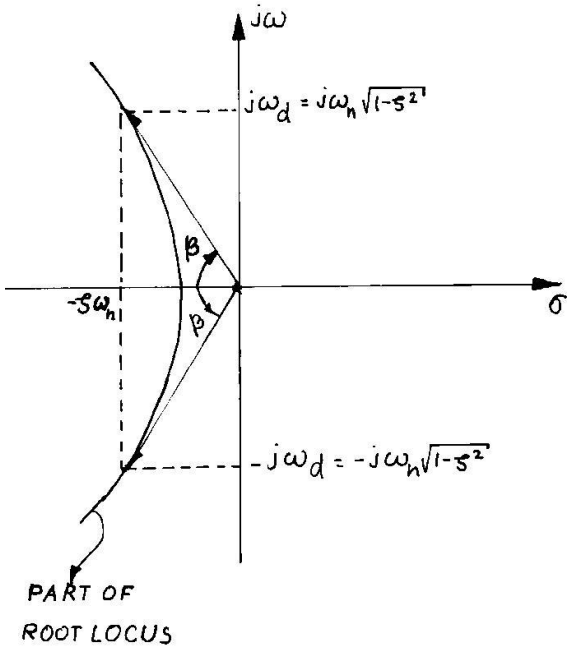
1.11. Gain Margin

$$GM \quad 20\log \frac{K_2}{K_1} \tag{10.69}$$

K_1 is the gain of a feedback system at some point on the root-locus

K_2 is the gain at which the system becomes unstable

1.12. Transient Characteristics



Where, $\tan \frac{\sqrt{1 \quad ^2}}{1}$

2. Percentage overshoot

$$M_p \quad e \quad / \tan \tag{10.70}$$

9. Settling time

$$t_s \frac{4}{n} \tag{10.71}$$

3.6Steady-state error is also related to K.

Example

Problem-1: Draw the root-locus of the feedback system whose open-loop transfer function is given by $G(s)H(s) \frac{K s^2 \quad 10 s \quad 100}{s^4 \quad 20 s^3 \quad 100 s^2 \quad 500 s \quad 1500}$, $H s \quad 1$

(a) Determine the value of gain at which the system will be stable and as well have a maximum overshoot of 5%.

(b) What is the gain margin at this point?

(c) What is the steady-state error for a unit step excitation at the above point?

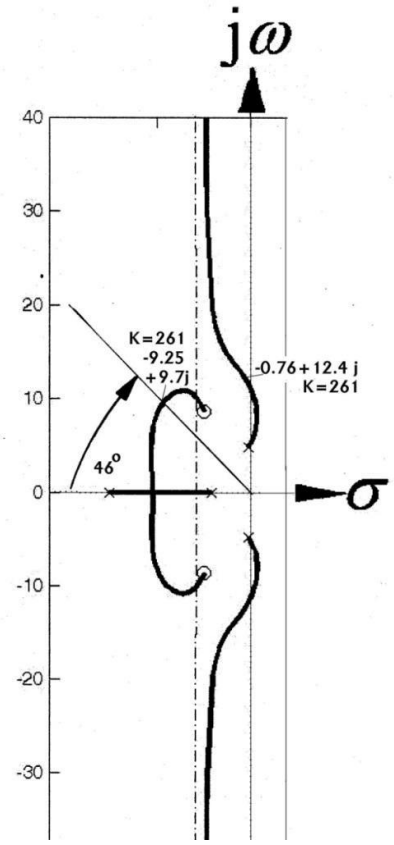
Solution:

$$\tan \quad 1.0487$$

(b) $\overline{\ln M_p}$

46^0

$$\frac{1}{\sqrt{1 \quad \tan^2}} 0.690 \qquad (10.72)$$



(b) $GM \; 20 \log \frac{192.2}{261} 2.65 dB$

(c) Position error

$$K_s \lim_{s \; 0 \; s} \frac{K(s^2 \quad 10 s \quad 100)}{^4 20 s \; ^3 \quad 100 s^2 \; 500 s \quad 1500 \quad 1500} \quad \frac{100K}{}$$

Steady-state error,

$$S_e(\;) \frac{1}{1 \quad K_s} \quad \frac{1}{1 \quad 100K / 1500} \quad \frac{1500}{1500 \quad 100K}$$

$$S_e(\;) \frac{1500}{1500 \quad 100 \quad 261} 1.104\%$$

a. Root Locus using MATLAB

Program 1: Draw the root locus for the following system $\frac{C s}{R s} \quad \frac{K}{s \; 1 \; s \; 2 \; s \; 4 \; s \; 5}$

Solution:

```
>> num=[01]
```

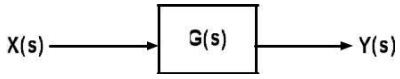
```
num=
0          1
>>q1=[1 1]; >> q2=[1
2]; >> q3=[1 3]; >>
q4=[1 4];
>>den=conv(q1,q2);
>> den=conv(den,q3);
>> den=conv(den,q4);
den=

1          12 49 78 40
>>sys=tf(num,den)
Transfer
  function: 1
--------------------
   s^4+12s^3+49s^2+78s+40
>>rlocus(sys)
```

11. Frequency Response Analysis

11.1. Frequency Response

This is defined as the steady-state response of a system due to a sinusoidal input.



Here,

$$G(s) = \frac{C(s)}{R(s)} = \frac{N(s)}{(s+a)(s+b)(s+c)\ldots} \tag{11.1}$$

$$C(s) = \frac{N(s)R(s)}{(s+a)(s+b)(s+c)\ldots} \tag{11.2}$$

Let, $r(t) = A\sin(\omega t)$, then

$$R(s) = \frac{A\omega}{s^2+\omega^2} \tag{11.3}$$

Using eq (3) in eq (2),

$$\begin{aligned} C(s) &= \frac{N(s)}{(s+a)(s+b)(s+c)\ldots(s+j\omega)(s-j\omega)} \\ C(s) &= \frac{A_1}{(s+a)} + \frac{A_1}{(s+b)} + \frac{A_1}{(s+c)} + \ldots + \frac{B_1}{(s+j\omega)} + \frac{B_2}{(s-j\omega)} \end{aligned} \tag{11.4}$$

In time domain, eq (5) becomes

$$c(t) = A_1 e^{-at} + A_2 e^{-bt} + A_3 e^{-ct} + \ldots + B_1 e^{-j\omega t} + B_2 e^{j\omega t} \tag{11.5}$$

The term with A_i terms are decaying components. So, they tend to zero as time tends to infinity. Then, eq (5) becomes

$$C_{ss}(t) = B_1 e^{-j\omega t} + B_2 e^{j\omega t} \tag{11.6}$$

Where,

$$\begin{aligned} B_1 &= \left.\frac{A\,G(s)}{(s+j\omega)}\right|_{s=-j\omega} = \frac{A}{2j}|G(-j\omega)|e^{-j\angle G(-j\omega)} \\ B_2 &= \left.\frac{A\,G(s)}{(s-j\omega)}\right|_{s=j\omega} = \frac{A}{2j}|G(j\omega)|e^{j\angle G(j\omega)} \end{aligned} \tag{11.7}$$

Since, $|G(-j\omega)| = |G(j\omega)|$ and $\angle G(-j\omega) = -\angle G(j\omega)$

$$c\,t \quad \frac{A}{2j}\left|G\ j\right| e^{j\,t} \quad \frac{A}{2j}\left|G\ j\right| \quad e^{j\,t} \tag{11.8}$$

$$c\,tA \quad \left|G\ j\right| e \quad \frac{j\,t\ e^{j} \quad e^{j}}{2\,j} \tag{11.9}$$

$$c\,t\ A \quad \left|\ G\ j\ \right|\sin\ t \tag{11.10}$$

$$c\,t\ \ B\ \ \sin\ t \tag{11.11}$$

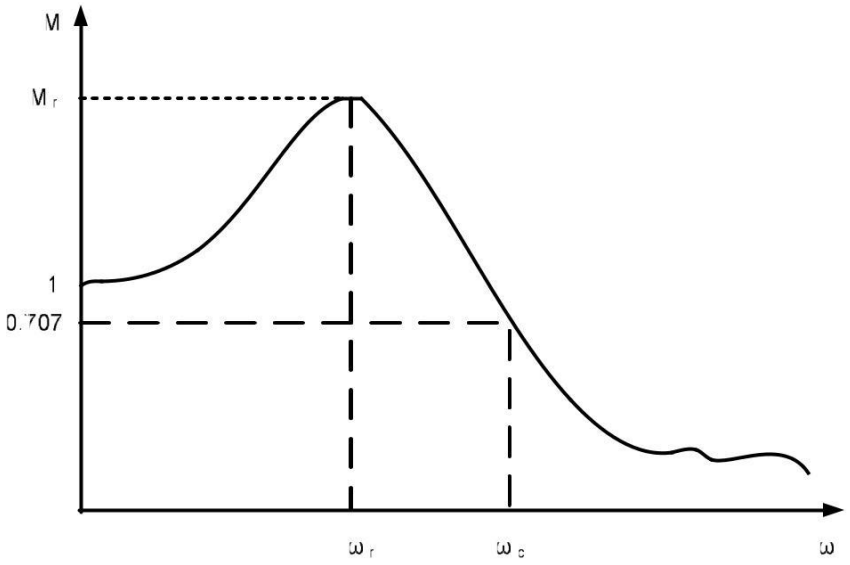
Where, $B \quad A\left|G\ j\right|$

Therefore, the steady-state response of the system for a sinusoidal input of magnitude A and frequency is a sinusoidal output with a magnitude B , frequency and phase shift .

The following plots are used in frequency response.

Polar plot
Bode plot
Magnitude versus phase angle plot

11.2. Definition of frequency domain specifications



(i) Resonant peak M_r : Maximum value of $M\ j$ when is varied from 0 to ∞.

(ii) Resonant frequency $_r$: The frequency at which M_r occurs

(iii) Cut-off frequency $_c$: The frequency at which $M\ j$ has a value $\frac{1}{\sqrt{2}}$. It is the frequency at which the magnitude is 3dB below its zero frequency value

(iv) Band-width $_b$: It is the range of frequencies in which the magnitude of a closed-loop system is $\frac{1}{\sqrt{2}}$ times of M_r

(v) Phase cross-over frequency: The frequency at which phase plot crosses -180^0

(vi) Gain margin (GM): It is the increase in open-loop gain in dB required to drive the closed-loop system to the verge of instability

(vii) Gain cross-over frequency: The frequency at which gain or magnitude plot crosses 0dB line

(viii) Phase margin (PM): It is the increase in open-loop phase shift in degree required to drive the closed-loop system to the verge of instability

11.3. Correlation between time and frequency response For a second order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{11.12}$$

Putting $s = j\omega$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + j2\zeta\omega_n}$$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right)} \tag{11.13}$$

Let, $u = \frac{\omega}{\omega_n}$, then

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{1 - u^2 + j2\zeta u} \tag{11.14}$$

Now,

$$M(j\omega) = |M(j\omega)| \angle M(j\omega) \tag{11.15}$$

Where,

$$|M(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}}$$

$$\tan^{-1}\frac{2\zeta u}{1-u^2} \tag{11.16}$$

Now,

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{11.17}$$

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} \tag{11.18}$$

$$_{b} \quad _{n}\sqrt{1\ 2^{\ 2}\ \ 4\sqrt[4]{4^{\ 2}\ \ \ \ 2}} \tag{11.19}$$

$$PM \quad 180^{0} \tag{11.20}$$

Where,$\tan^{\ 1}$ $\dfrac{2}{\sqrt{\sqrt{4^{\ 2}\ \ 1\ 2^{\ 2}}}}$

11.4. Advantages

Good accuracy

Possible to test in lab

Can be used to obtain transfer function that is not possible with analytical techniques

Easy to design open-loop transfer function from closed-loop performance in frequency domain

It is very easy to visualize the effect of disturbance and parameter variations.

11.1.10 Disadvantages

Applied only to linear systems

Frequency response for existing system is possible to obtain if the time constant is up to few minutes

Time consuming procedure

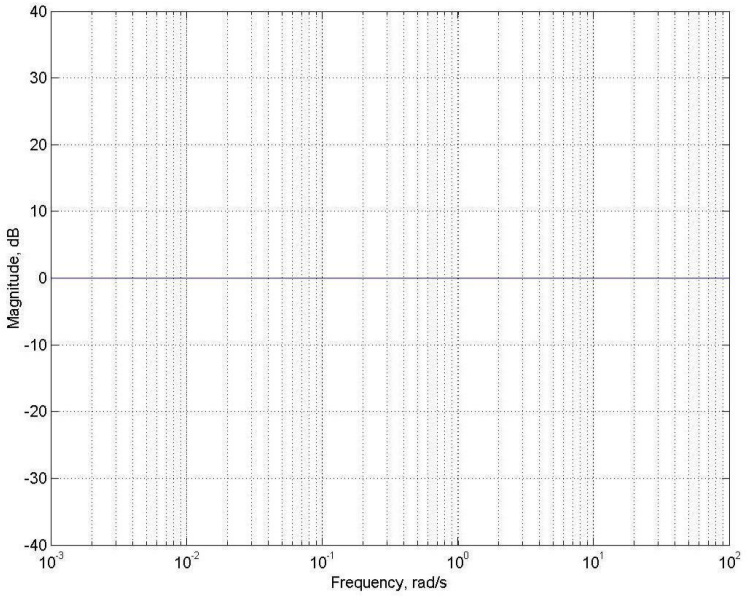
Old and back dated method

Equation Chapter 12 Section 1

12. Bode Plots

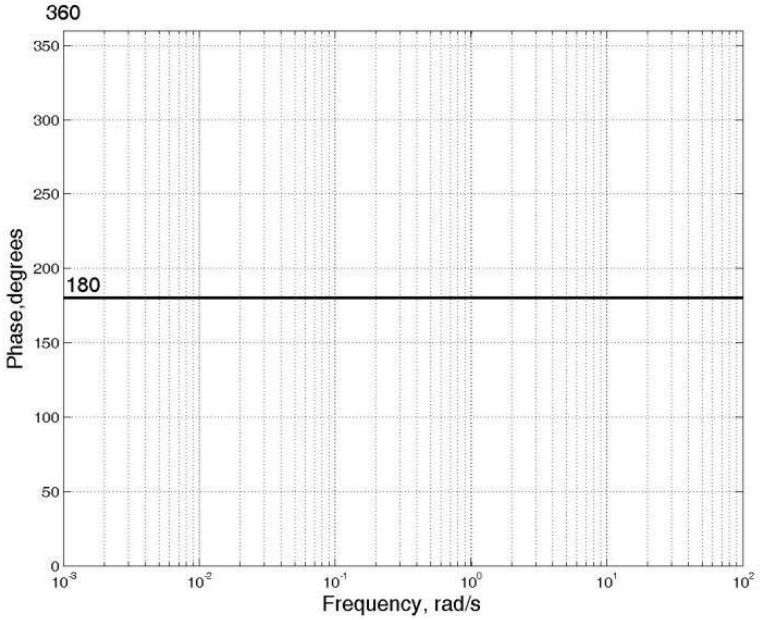
12.1. Magnitude plot and phase plot on a semi-log paper

Magnitude plot on a semi-log paper



$$M \quad 20\log|G(j)H(j)|\ \text{dB}$$

Phase plot on a semi-log paper



12.2. Magnitude versus phase Bode plot Nichols plot

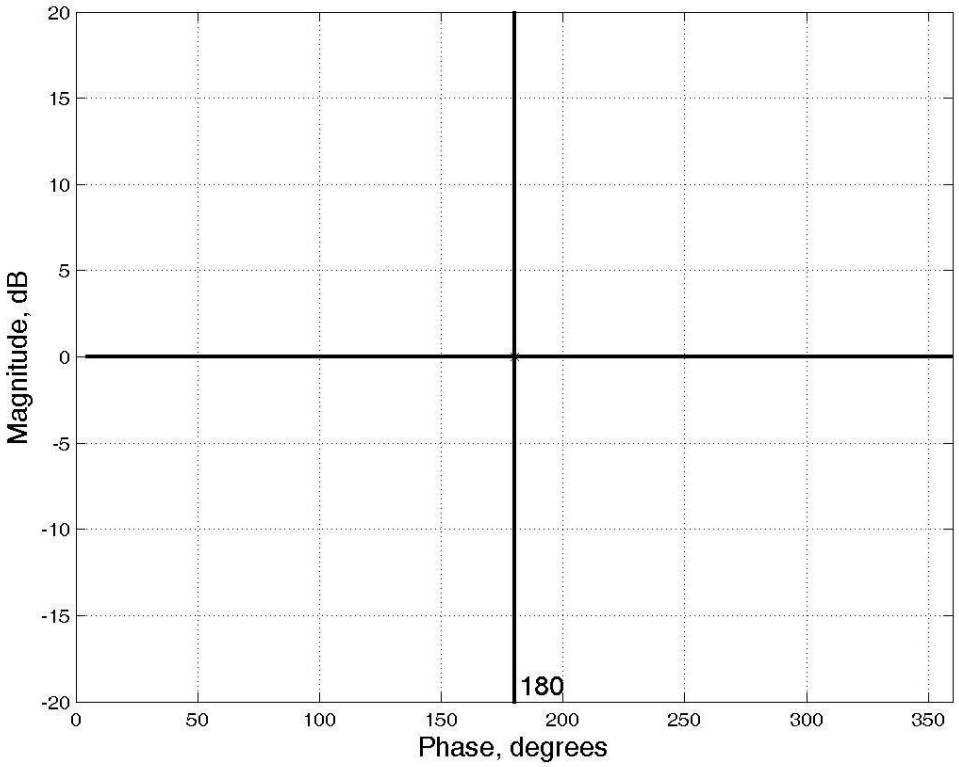


Table 12.1 Basic frequency response factors

No	Laplace term	Frequency response	Type of factor
1	K	K	Constant
2	s	j	Derivative factor
3	1/s	$1/j$	Integral factor
4	s+1	$(1+j)$	First order derivative factor
5	1/(s+1)	$1/(1+j)$	First order integral factor
6	$s^2 \quad 2 \quad _n s \quad _n^2$	$_n^2 \quad ^2 \quad j2 \quad _n$	Second order derivative factor
7	$\frac{1}{s^2 \quad 2 \quad _n s \quad _n^2}$	$\frac{1}{_n^2 \quad ^2 \quad j2 \quad _n}$	Second order integral factor

12.3. Derivative factor: magnitude

$$M \ 20 \log \ | j \ | 20\log \ \text{dB} \tag{12.1}$$

$$j \ 90^{0} \tag{12.2}$$

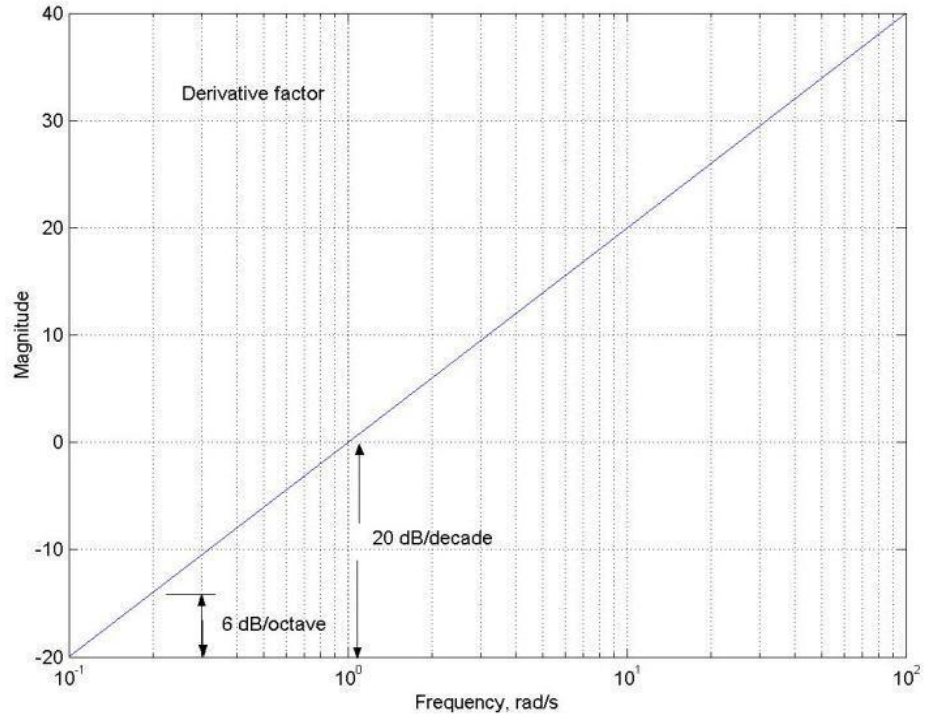
$$M \ 20\log \quad 20\log 20\log \quad \frac{2}{1} \text{ dB/decade} \tag{12.3}$$

$$M \ 20\log 10 \ 20 \text{ dB/decade} \tag{12.4}$$

$$M \ 20 \log 2 \ 6 \text{ dB/octave} \tag{12.5}$$

Table 12.2 Magnitude variation of a derivative factor for various multiples of the initial frequency

$\frac{2}{1}$	1	2	3	4	5	6	7	8	9	10
M dB	0	6	10	12	14	16	17	18	19	20



12.4. Derivative Factor: (phase)

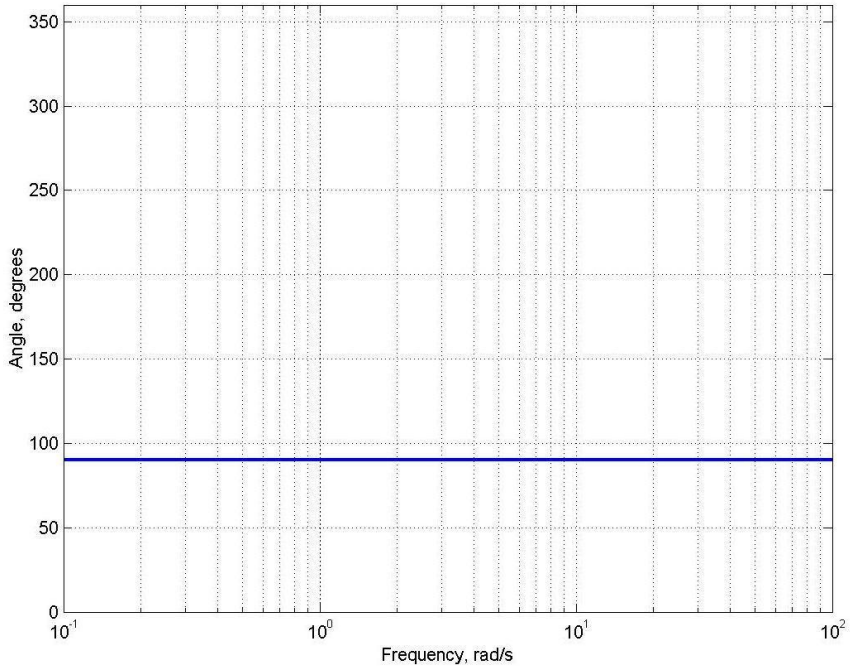


Table 11.103Derivative factor

	Frequency, rad/s				
	0.1	1	10	30	100
Magnitude, dB	-20	0	20	30	40
Phase, degrees	90	90	90	90	90

12.1.10 Integral factor: magnitude

$$M \quad 20\log\left|\frac{1}{}\right| \quad 20\log \quad \text{dB} \tag{12.6}$$

$$j \quad 270^{0} \tag{12.7}$$

$$M \qquad 20\log_{2}\ 20\log_{1}\ 20\log^{2}\ \text{dB/decade}_{1} \tag{12.8}$$

$$M \quad 20\log 10 \quad 20\ \text{dB/decade} \tag{12.9}$$

$$M \quad 20\log 2 \quad 6\ \text{dB/octave} \tag{12.10}$$

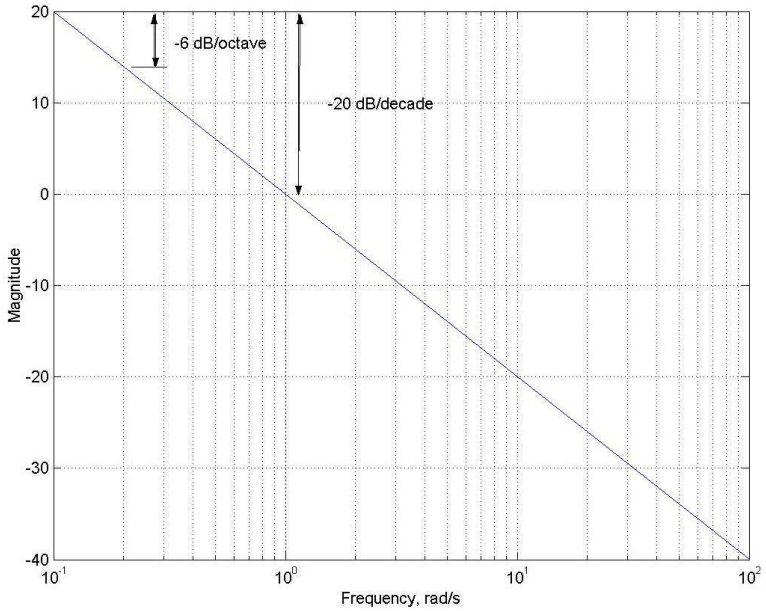


Table 12.4 Magnitude variation of an integral factor for various multiples of the initial frequency

$\frac{2}{1}$	1	2	3	4	5	6	7	8	9	10
M , dB	0	-6	-10	-12	-14	-16	-17	-18	-19	-20

12.1.11. Integral factor: phase

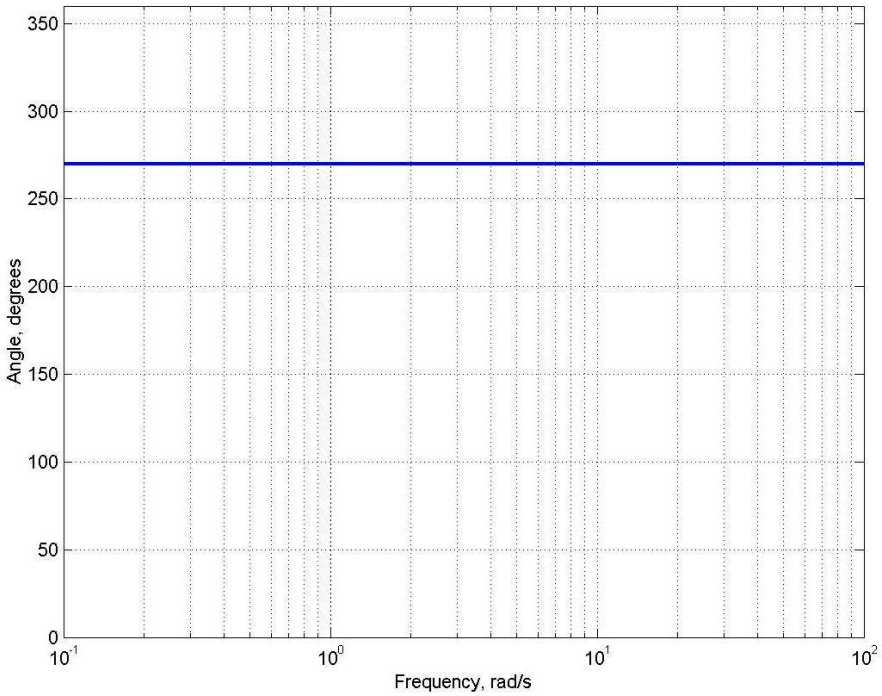


Table 12.5 Bode magnitude and phase of an integral factor

	Frequency, rad/s				
	0.1	1	10	20	100
Magnitude, dB	20	0	-20	-26	-40
Phase, degrees	270	270	270	270	270

12.1.12. First-order derivative factor: magnitude

$$M \ 20 \log |1 \ j \quad |20\log(\sqrt{1 \quad }_{2} \text{ dB} \tag{12.11}$$

For << , M≈0 dB
c

For >> ,
c

$$M \ 20\log \ __ \text{dB} \tag{12.12}$$
$$c$$

Here, =1/ = corner frequency
c

For >
c

$$M \ 20\log \qquad 20\log 20\log \frac{2}{1} \tag{12.13}$$
$$2 \qquad 1 \qquad 1$$

$$M \ 20\log 10 \ 20 \text{ dB/decade} \tag{12.14}$$

$$M \ 20 \log 2 \ 6 \text{ dB/octave} \tag{12.15}$$

Table 12.6 Magnitude variation of a first-order derivative factor for various multiples of the corner frequency

— c	1	2	3	4	5	6	7	8	9	10
M, dB	0	6	10	12	14	16	17	18	19	20

12.2. First-order derivative factor: phase

$$1 \ j\arctan \tag{12.16}$$

$$\begin{cases} 0 & ; w \ \frac{w_c}{10} \\ 45^0 \ 1 \log \ __ & ; \frac{w_c}{10} \ w \ 10w_c \\ 90 & ; w \ 10w_c \end{cases} \tag{12.17}$$
$$c$$

Table 12.7 Phase angles of a first-order derivative factor around the corner frequency

$\frac{}{c}$	1	2	3	4	5	6	7	8	9	10
, deg	45	59	66	72	76	80	83	86	88	90
$\frac{}{c}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
, deg	0	2	4	7	10	14	18	24	31	45

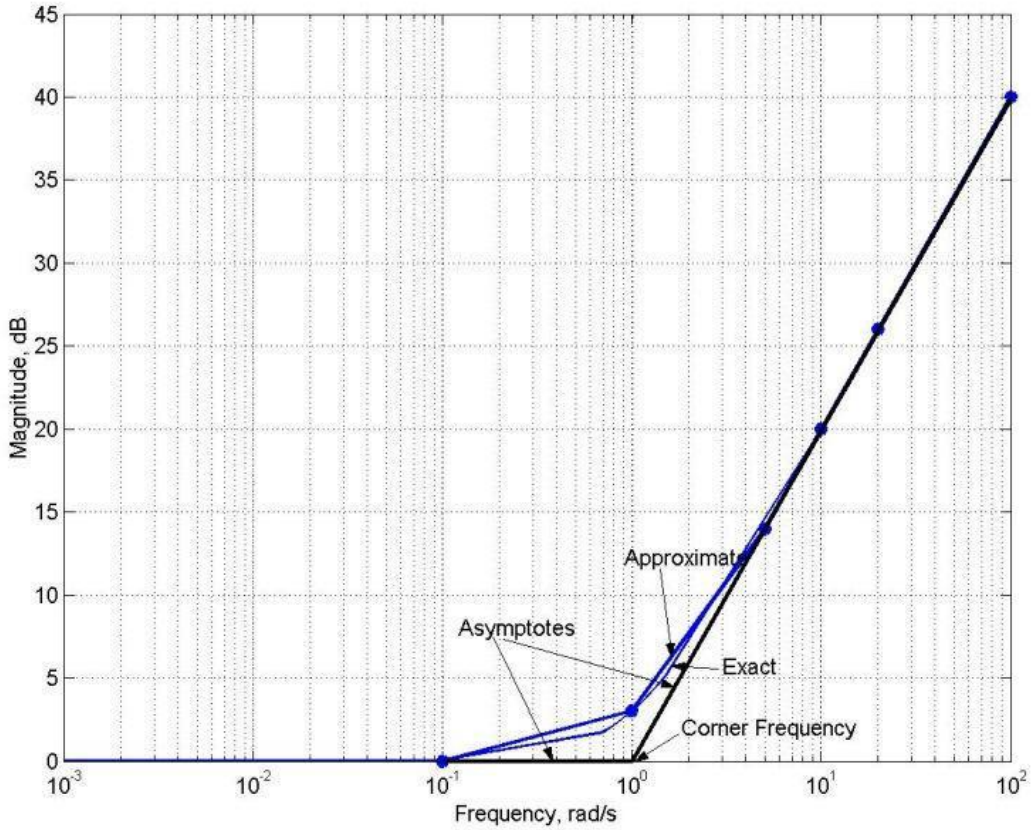
12.9. First-order derivative factor

For 1

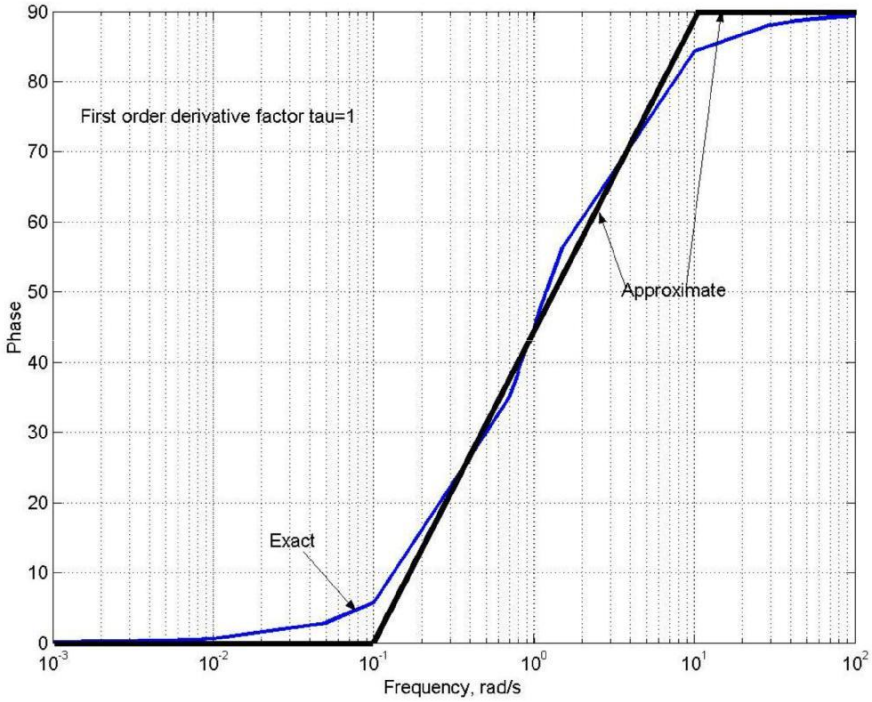
Table 11.108 Bode magnitude and phase

	Frequency, rad/s					
	0.1	1	5	10	20	100
Magnitude, dB	0	3	14	20	26	40
Phase, degrees	0	45	76	90	90	90

First-order derivative factor: magnitude (3 dB correction at the corner frequency)



First-order derivative Factor: phase



12.3.6 First-order integral factor: magnitude

$$M \quad 20\log \left| \frac{1}{1 \quad j} \right| 20\log \quad \frac{1}{\sqrt{\quad 1}} \quad \text{dB} \qquad (12.18)$$

$$M \quad 0, \qquad w \quad w_c$$
$$M \quad 20\log \quad __ dB \,, \quad w \quad w \qquad (12.19)$$
$$c \qquad c$$

$$M \qquad 20\log_{\ 2}\ 20\log_{\ 1}\ 20\log^{\ 2}\ \text{dB/decade}\ 1 \qquad (12.20)$$

$$M \qquad 20\log 2 \qquad 6\ \text{dB/octave} \qquad (12.21)$$

Table 12.9 Magnitude variation of a first-order integral factor for various multiples of the corner frequency

__ / c	1	2	3	4	5	6	7	8	9	10
M, dB	0			-6	-10	-12	-14	-16	-17	-18	-19	-20

Table 12.10 Phase angles of a first-order integral factor around the corner frequency

$\frac{}{c}$	1	2	3	4	5	6	7	8	9	10
deg	315	301	294	288	284	280	277	274	272	270
$\frac{}{c}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
deg	360	358	356	353	350	346	342	336	329	315

First-order integral factor: phase

$$=360, \ < \ _{c}/10$$

$$360 \ 45 \ 1 \ \log \ \frac{0}{} \ \frac{}{} \ , \ _{c}/10< \ <10 \ _{c}$$

$$\frac{}{c}$$

$$360 \ 45 \ \frac{0}{} \ 1 \ \log \frac{}{}$$

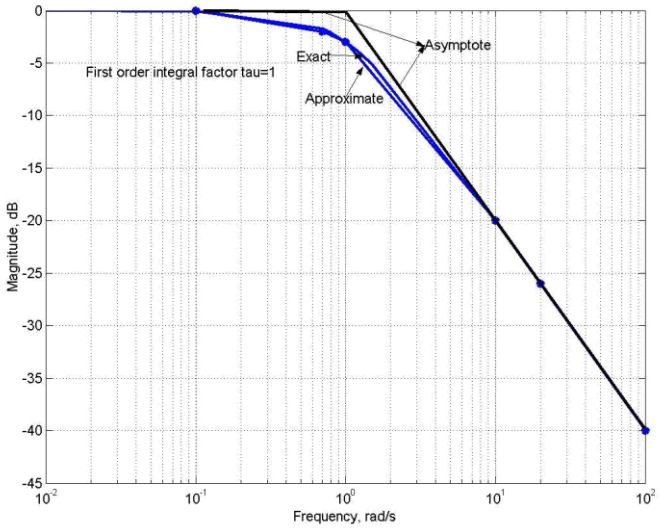
$$\frac{}{c}$$

$$=270, \ >10 \ _{c}$$

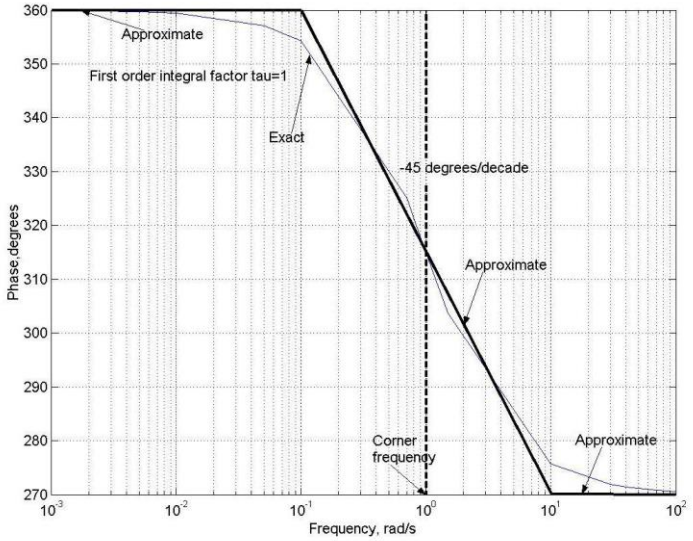
Table 12.11 Bode magnitude and phase of a first-order integral factor

	Frequency, rad/s							
	0.01	0.1	0.7	1	7	10	20	100
Magnitude, dB	0	0	-2	-3	-17	-20	-26	-40
Phase, degrees	360	360	322	315	277	270	270	270

First-order integral factor: magnitude



First-order integral factor: phase



12.12. Second-order derivative factor: magnitude

$$M \; 20 \log | \; {}_n^{2} \quad {}^{2} \quad j2 \quad {}_n \; |$$

$$20 \log \quad {}_n^{2} \quad \sqrt{1 \quad \frac{}{{}^{2}} \quad 2 \quad \frac{}{}} \qquad (12.22)$$

$$n \qquad\qquad n$$

$$\mathbf{M \;\; 40 log w_n \, , w \;\; w_n}$$

$$\mathbf{M \;\; 20 log \, (2 z w_n \;\; {}^{2} \,), \; w \;\; w_n} \qquad (12.23)$$

$$\mathbf{M \;\; 40 \; log w, w \;\; w_n}$$

For $w \qquad w_n$

$$M \quad 40\log \qquad 40\log 40\log \frac{2}{1} \text{ dB/decade} \tag{12.24}$$

$$M \quad 40\log 10 \quad 40 \text{ dB/decade} \tag{12.25}$$

$$M \quad 40\log 2 \quad 12 \text{ dB/octave} \tag{12.26}$$

Magnitude variation of a second-order derivative factor for various multiples of the resonant frequency

$\frac{}{n}$	1	2	3	4	5	6	7	8	9	10
M dB	0	12	20	24	28	32	34	36	38	40

Second-order derivative factor: phase

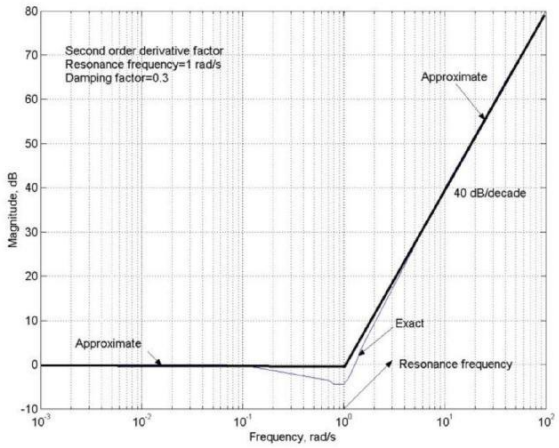
$$| \quad ^2 \qquad ^2 \qquad j2| \ \arctan \frac{2\frac{}{n}}{1\frac{}{^2}} \tag{12.27}$$

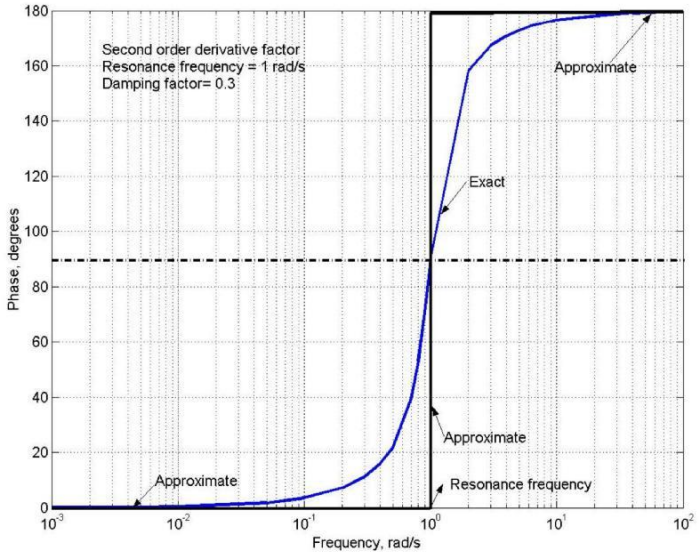
$$\begin{aligned} 0^0, \quad & w \quad \frac{w_n}{10} \\ 90^0, \quad & w \quad w_n \\ 180^0, \quad & w \quad 10w_n \end{aligned} \tag{12.28}$$

Bode magnitude and phase

ω_n=1 rad/s, ζ=0.3

Frequency, rad/s	0.01	0.1	0.7	1	3	10	100
Magnitude, dB	0	0	-4	-4	18	40	80
Phase, degrees	0	0	39	90	167	180	180





Second-order integral factor

$$M \quad 20\log \left| \frac{1}{n^2 \quad 2 \quad j2 \quad n} \right| \qquad dB \quad 20\log \frac{1}{\sqrt{1 \quad \frac{2}{n}^2 \quad 2 \quad \frac{2}{n}^2}} \quad dB \qquad (12.29)$$

$$M \approx -40\log_{10} \quad , \ll \quad n$$

$$M = -20\log_{10}(2 \quad n^2), \quad = \quad n$$

$$M = -40\log_{10} \quad , \gg \quad n$$

$$M \quad 40\log \quad 40\log 40\log \quad \frac{2}{1} \quad dB / decade \qquad (12.30)$$

$$M \quad 40\log 10dB \quad 40dB \tag{12.31}$$

Magnitude variation of a second-order integral factor for various multiples of the resonant frequency

$\frac{}{n}$	1	2	3	4	5	6	7	8	9	10
M, dB	0	-12	-20	-24	-28	-32	-34	-36	-38	-40

$$\left| \frac{1}{{}^{2}_{n} \quad {}^{2} \quad j2 \quad {}_{n}} \right| \quad 360 \ \arctan \frac{2 \ \frac{}{n}}{1 \quad \frac{}{{}^{2}_{n}}} \tag{12.32}$$

$=0, \ < {}_{n}$

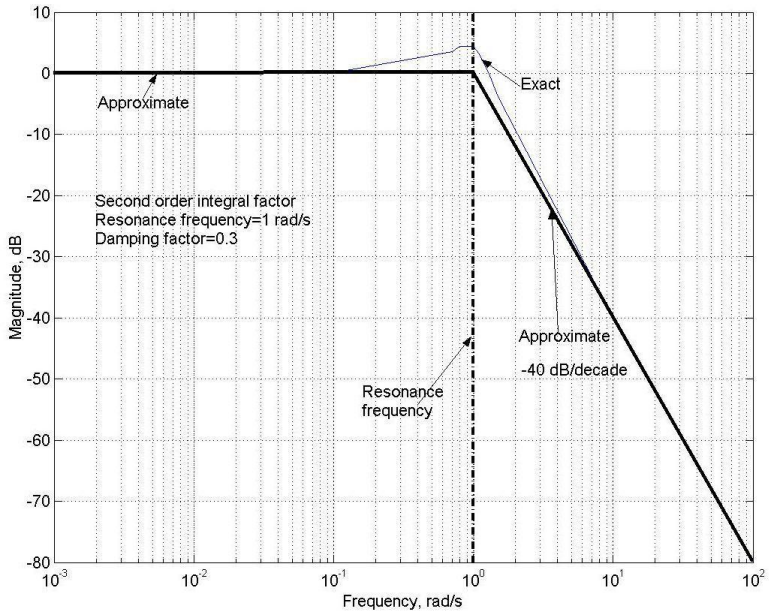
$=270^{0}, \ = {}_{n}$

$=180^{0}, \ > {}_{n}$

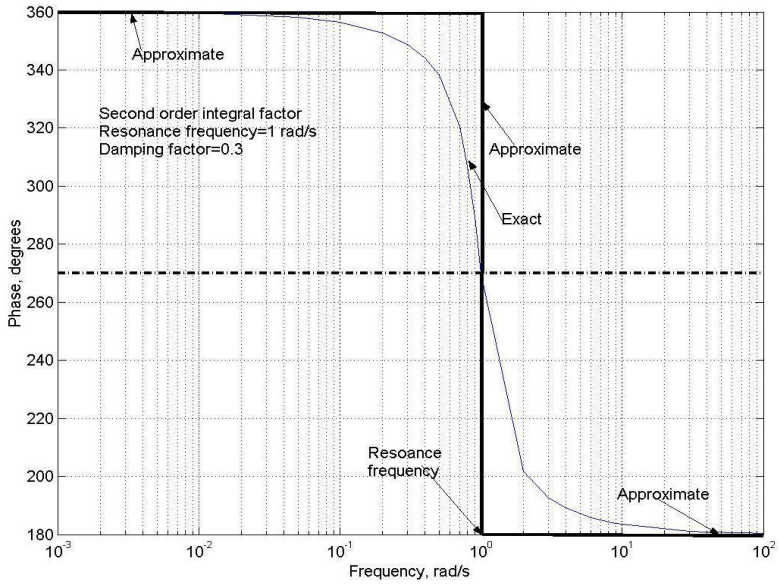
Bode magnitude and phase

Frequency, rad/s	0.01	0.1	0.7	1	3	10	100
Magnitude, dB	0	0	4	4	-18	-40	-80
Phase, degrees	360	360	321	270	193	180	180

Magnitude plot



Phase plot



Example 11.6

Draw the Bode magnitude and phase plot of the following open-loop transfer function and determine gain margin, phase margin and absolute stability?

$$G(s)H(s)\frac{1}{s(s\quad 1)}$$

Solution

Applying $s \quad j \quad ,$

$$G(j\ )H(j\ )\ \frac{1}{j\ (j\quad 1)}$$

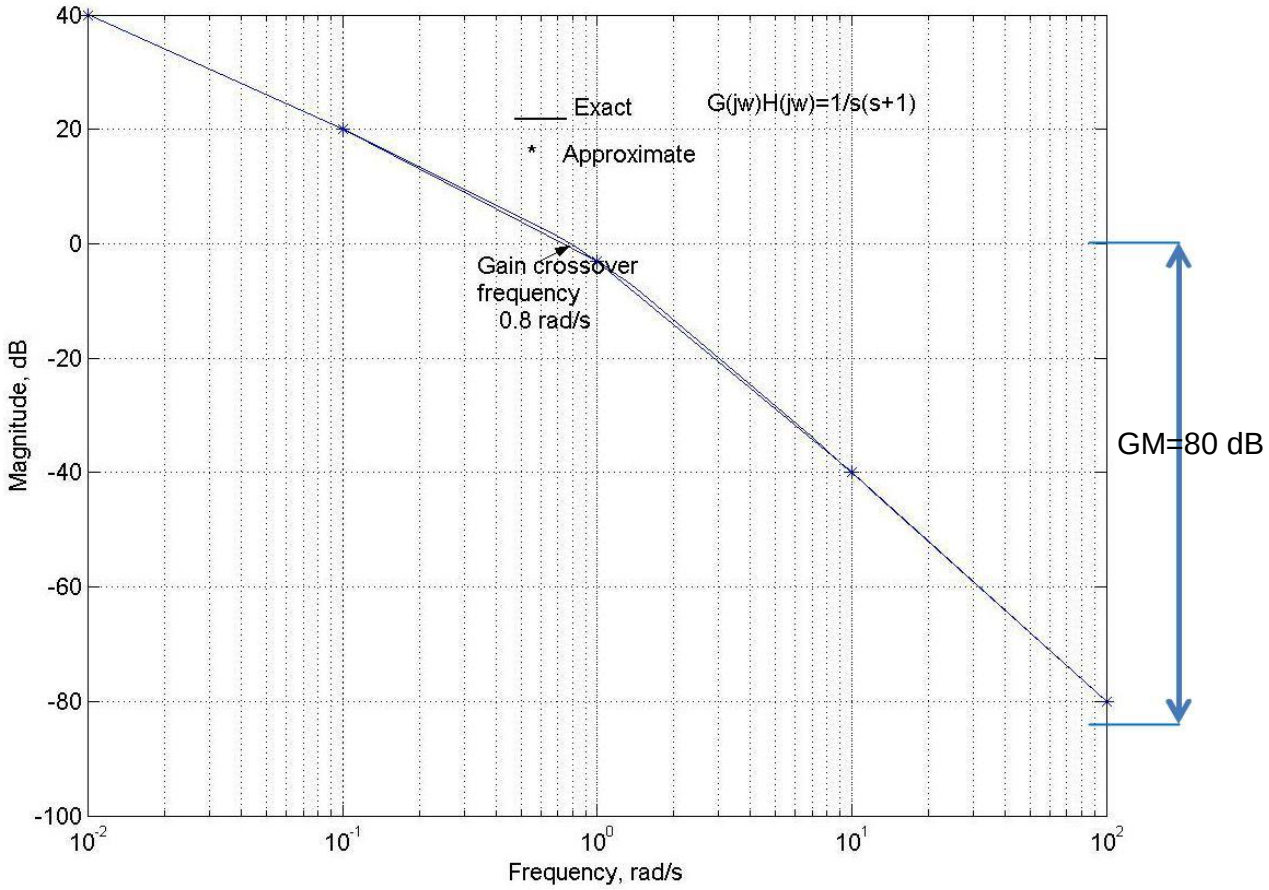
The above frequency response function has two factors: (1) Integral factor and (2) First order integral factor with a corner frequency of 1 rad/s

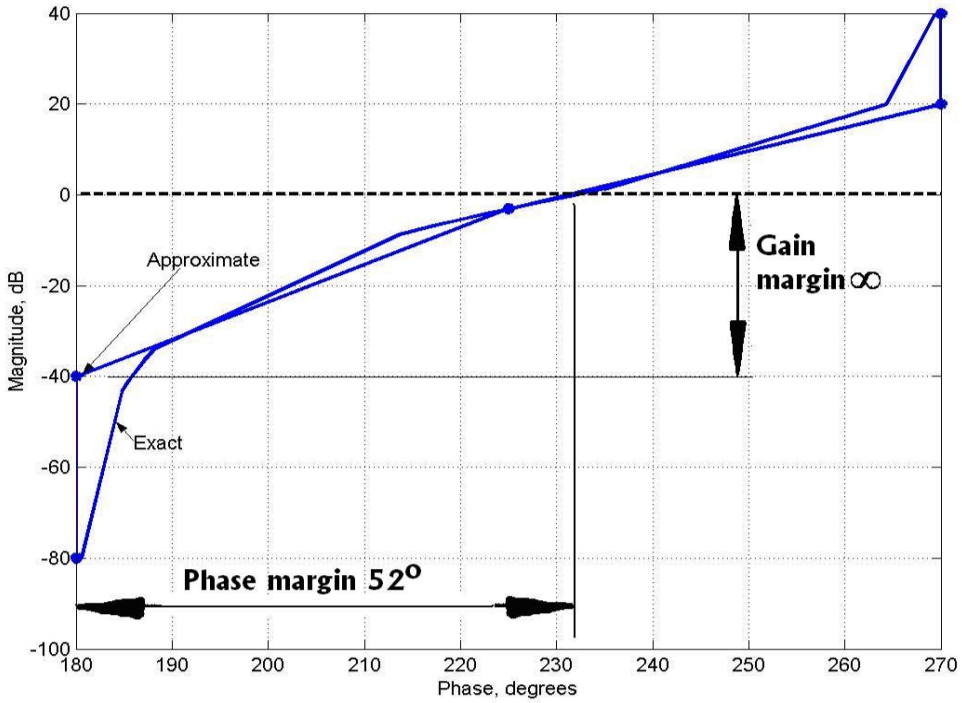
Bode magnitude of the transfer function

	Frequency, radians/s				
	0.01	0.1	1	10	100
$20\log\frac{1}{j}$ dB	40	20	0	-20	-40
$20\log\frac{1}{j\ 1}$ dB	0	0	-3	-20	-40
Magnitude, dB	40	20	-3	-40	-80

$_p$= 100 rad/s

	Frequency, rad/s				
	0.01	0.1	1	10	100
$\frac{1}{j}$ degrees	270	270	270	270	270
$\frac{1}{j\ 1}$ degrees	360	360	315	270	270
Bode phase, degrees	270	270	225	180	180


Exact
G(jw)H(jw)=1/s(s+1)
Approximate
Gain crossover
frequency
0.8 rad/s
GM=80 dB
Magnitude, dB
Frequency, rad/s
40
20
0
-20
-40
-60
-80
-100
10^-2
10^-1
10^0
10^1
10^2


Approximate
Exact
Gain
margin ∞
Phase margin 52°
Magnitude, dB
Phase, degrees
40
20
0
-20
-40
-60
-80
-100
180
190
200
210
220
230
240
250
260
270

Example 11.7

Draw the Bode magnitude and phase plot of the following open-loop transfer function and determine gain margin, phase margin and absolute stability?

$$G(s)H(s)\frac{1}{s(s\ 2)s\ 4)}$$

Solution

$$G(j\ )H(j\ )\ \frac{1}{8\,j\frac{j}{2}\quad 1\frac{j}{4}\quad 1}$$

The corner frequencies corresponding to first order integral factors are 2 rad/s and 4 rad/s. Minimum frequency is chosen as 0.01 rad/s and maximum frequency 100 rad/s.

Table 11.6 Computation of Bode magnitude using asymptotic properties of the integral first-order term $\frac{1}{2}$

	x1	x2		x1	x10		x2	x1		x1	x2		x1	x10
Frequency, rad/s	2	4		2	20		20	10		20	40		10	100
Magnitude, dB	0	-6		0	-20		-20	-14		-20	-26		-14	-34

Table 11.7 Computation of Bode magnitude using asymptotic properties of the integral first-order term $\frac{1}{4}$

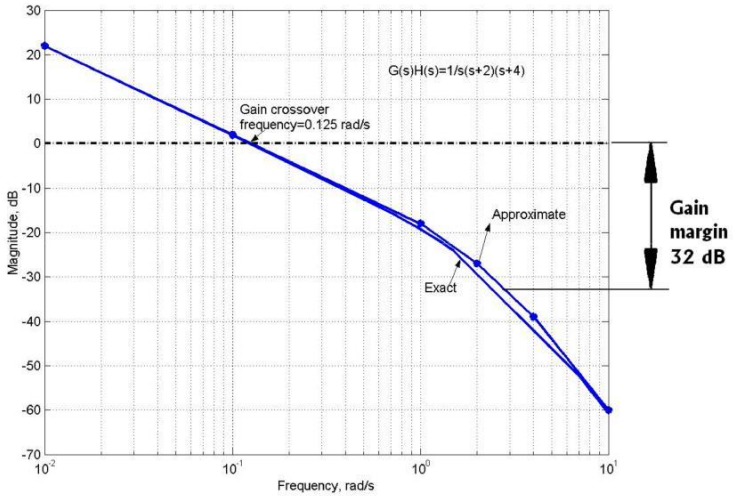
	x1	x10		x2	x1		x2	x1		x1	x10
Frequency, rad/s	4	40		40	20		20	10		10	100
Magnitude, dB	0	-20		-20	-14		-14	-8		-8	-28

Table 12.3 Bode magnitude

	Frequency, rad/s										
Factor	0.01	0.1	0.2	0.4	1	2	4	10	20	40	100
$20\log\frac{1}{8}$	-18	-18	-18	-18	-18	-18	-18	-18	-18	-18	-18
$20\log\frac{1}{j}$	40	20	14	8	0	-6	-12	-20	-26	-32	-40
$20\log\frac{1}{\frac{j}{2}\ 1}$	0	0	0	0	-1	-3	-6	-14	-20	-26	-34
$20\log\frac{1}{\frac{j}{4}\ 1}$	0	0	0	0	0	-1	-3	-8	-14	-20	-28
Bode magnitude,	22	2	-4	-10	-18	-28	-39	-60	-78	-96	-120

dB											

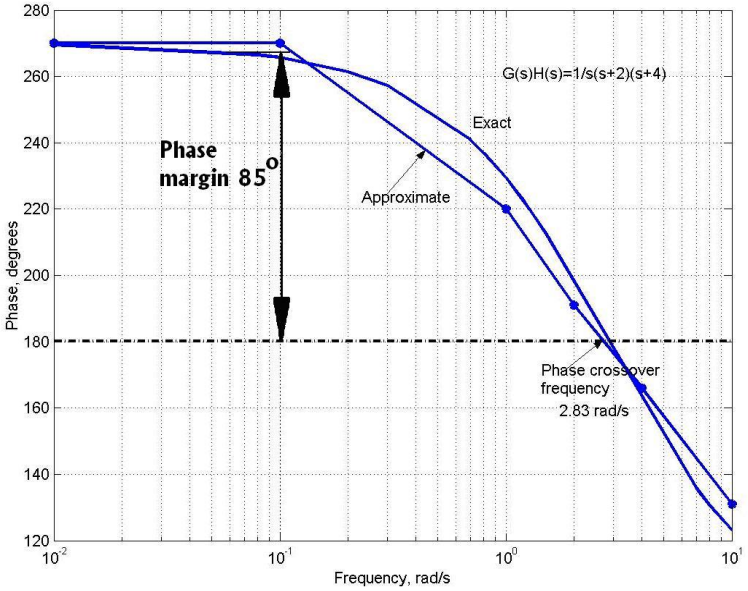
Bode magnitude



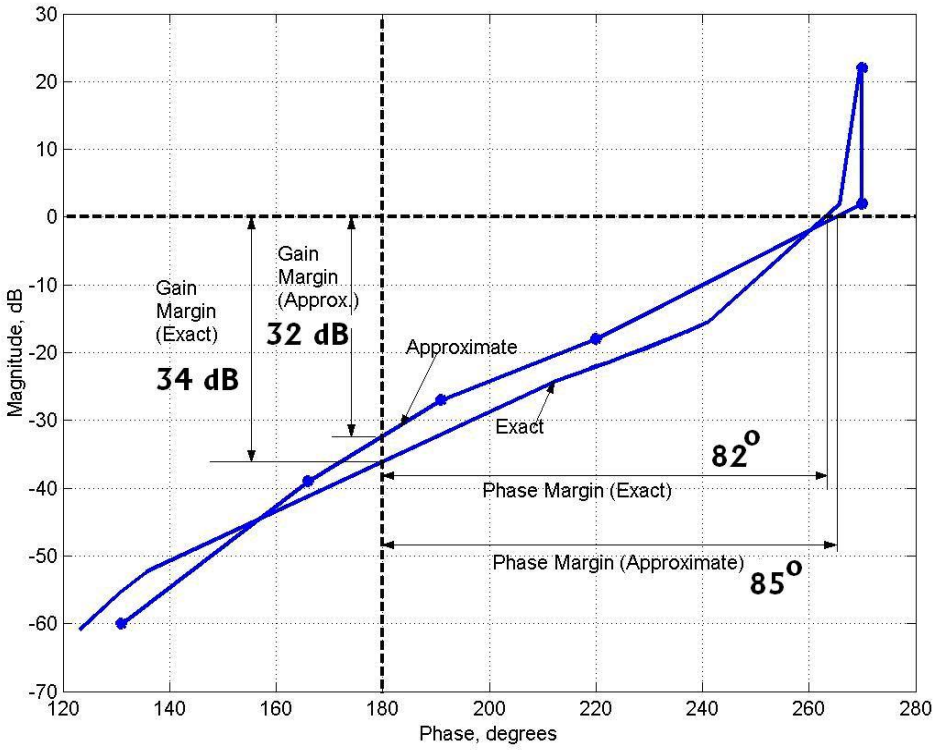
Bode phase

	Frequency, rad/s										
Factor	0.01	0.1	0.2	0.4	1	2	4	10	20	40	100
$\frac{1}{8}$	0	0	0	0	0	0	0	0	0	0	0
$\frac{1}{j}$	270	270	270	270	270	270	270	270	270	270	270
$\frac{1}{\frac{j}{2}+1}$	360	360	360	346	328	315	301	284	270	270	270
$\frac{1}{\frac{j}{4}+1}$	360	360	360	360	342	326	315	297	285	270	270
Phase degrees	270	270	270	256	220	191	166	131	105	90	90

Phase plot


G(s)H(s)=1/s(s+2)(s+4)
Exact
Approximate
Phase
margin 85°
Phase crossover
frequency
2.83 rad/s
Phase, degrees
Frequency, rad/s
280
260
240
220
200
180
160
140
120
10^-2
10^-1
10^0
10^1

Bode plot



Example 12.1

Draw the Bode magnitude and phase plot of the following open-loop transfer function and determine gain margin, phase margin and absolute stability?

$$G(s)H(s)\frac{1}{s^2(s\quad 1)}$$

Solution

$$G(j\ )H(j\ )\quad\frac{1}{(j\ )(j\ )(j\quad 1)}$$

There are two integral factors and an integral first-order term with a corner frequency of 1 rad/s Bode magnitude

	Frequency, rad/s				
	0.01	0.1	1	10	100
$20\log\frac{1}{j}$ dB	40	20	0	-20	-40
$20\log\frac{1}{j}$ dB	40	20	0	-20	-40

$20\log\frac{1}{j\ 1}$ dB	0	0	-3	-20	-40
Bode magnitude, dB	80	40	-3	-60	-120

Example 12.2

Draw the Bode magnitude and phase plot of the following open-loop transfer function and determine gain margin, phase margin and absolute stability?

$$G(s)H(s) \quad \frac{1}{s^4\ 5s^3\ 8s^2\ 6s}$$

Solution

$$G(s)H(s)\frac{1}{s(s^2\ 2s\ 2)(s\ 3)}$$

$$G(j\ )H(j\ ) \quad \frac{1}{j\ (j\ )^2\ 2(j\ )\ 2((j\ )\ 3)}$$

$$G(j\ )H(j\ ) \quad \frac{\frac{1}{3}}{j\ \ (2\ \ ^2)\ j2\ )\ 2\ (j\ \ \ \frac{}{3}1)}$$

Comparing the second order term with a standard second order term,

$$_n^{2\ \ 2}\ \ j2_n$$

$$2\ \text{and}\ \frac{1}{\sqrt{\ }}.$$

$$\overline{\sqrt{}}2$$

For the first order integral factor, $_c$=3 rad/s

For ζ> 0.5, the response at resonance is less than theresponse at frequencies less than the resonant frequencies

Table Computation of Bode magnitude using asymptotic properties of the integral second-order term

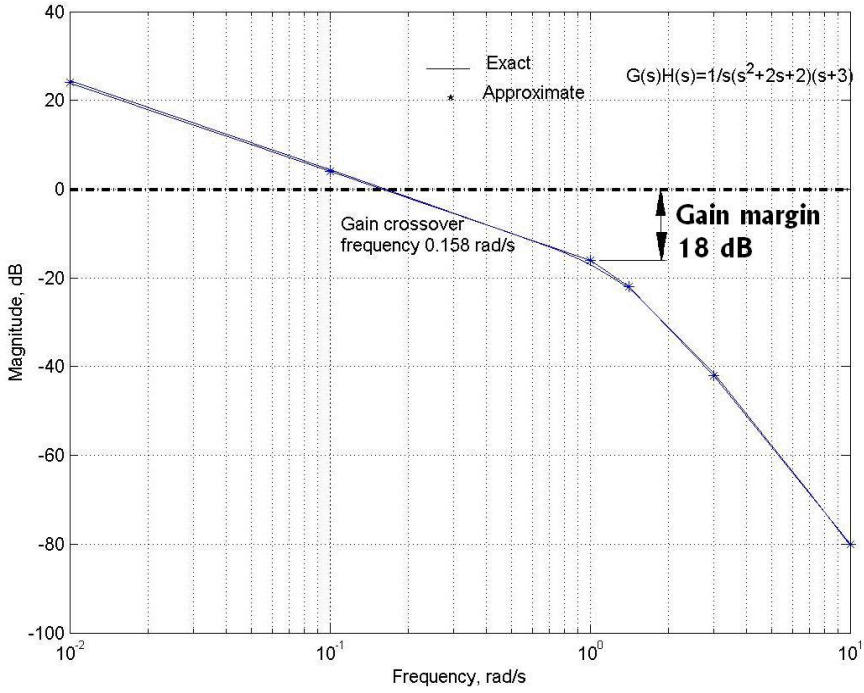
	x1	x10		x1	x2		x3	x1		x1	x10		x3	x1
Frequency, rad/s	1.4	14		14	30		30	10		10	100		30	3
Magnitude, dB	-6	-46		-46	-58		-58	-38		-38	-78		-58	-18

Table Computation of Bode magnitude using asymptotic properties of the integral first-order term

	x1	x3		x2	x1		x3	x1		x1	x10
Frequency, rad/s	3	30		30	14		30	10		10	100
Magnitude, dB	0	-20		-20	-14		-20	-10		-10	-30

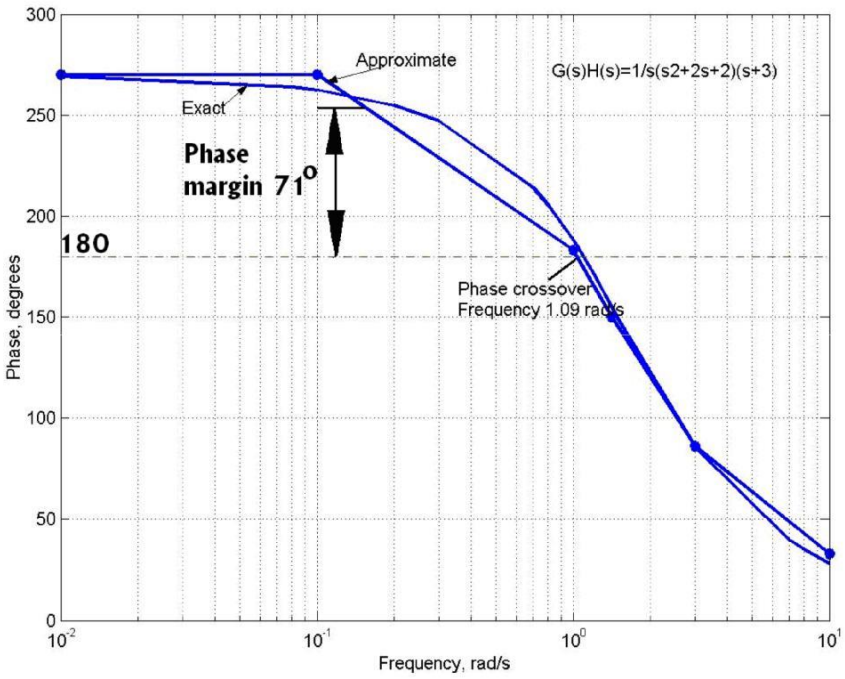
Bode magnitude

	Frequency, rad/s										
						n	c				
	0.01	0.1	0.14	0.3	1	$\sqrt{2}$	3	10	14	30	100
$20\log\frac{1}{3}$	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10	-10
$20\log\frac{1}{j}$	40	20	17	10	0	-3	-10	-20	-23	-30	-40
$20\log\frac{1}{(2\ ^2)\ j(2\ )}$	-6	-6	-6	-6	-6	-9	-18	-38	-46	-58	-78
$20\log\frac{1}{j\frac{}{3}\ 1}$	0	0	0	0	0	-1	-3	-10	-14	-20	-30
Bode magnitude, dB	24	4	1	-6	-16	-23	-41	-78	-93	-118	-158

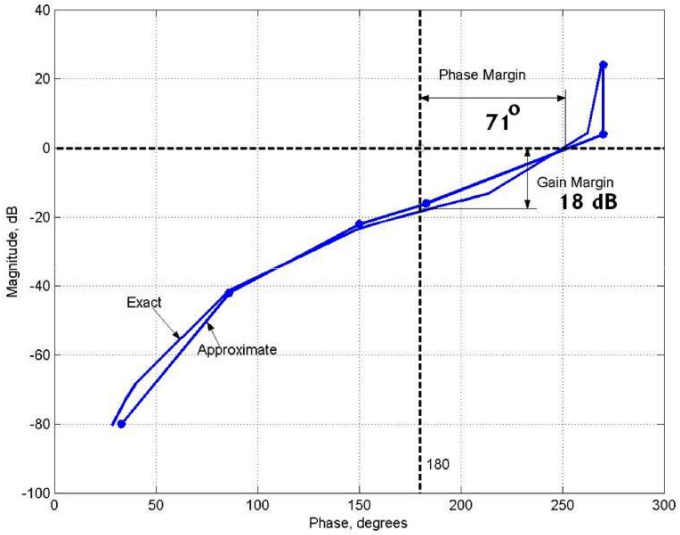


Bode phase

	Frequency, rad/s										
						n	c				
	0.01	0.1	0.14	0.3	1	$\sqrt{2}$	3	10	14	30	100
$\frac{1}{3}$	0	0	0	0	0	0	0	0	0	0	0
$\frac{1}{j}$ degrees	270	270	270	270	270	270	270	270	270	270	270
$\frac{1}{(2\ ^2)\ j(2\ )}$ degrees	360	360	360	343	297	270	221	192	180	180	180
$\frac{1}{j\overline{3}\ 1}$, degrees	360	360	360	360	336	330	315	291	285	270	270
Bode phase, degrees	270	270	250	253	183	150	86	33	15	0	0



Nichols plot



13. Polar Plots

It is a graphical method of determining stability of feedback control systems by using the polar plot of their open-loop transfer functions.

Example 13.1

Draw a polar plot of the open-loop transfer functionfor

$$G(s)H(s) \quad \frac{K}{s(s\ 1)} \tag{11.83}$$

Frequency response

$$G(j\ )H(j\ ) \quad \frac{K}{j\ (j\ \ 1)} \tag{11.84}$$

Magnitude

$$\left|G(j\ )H(\ )\right| \quad \frac{K}{1\sqrt{\ ^{2}}} \tag{11.85}$$

Angle

$$G(j\ )H(j\ ) \quad \frac{}{2}\tan^{1} \tag{11.86}$$

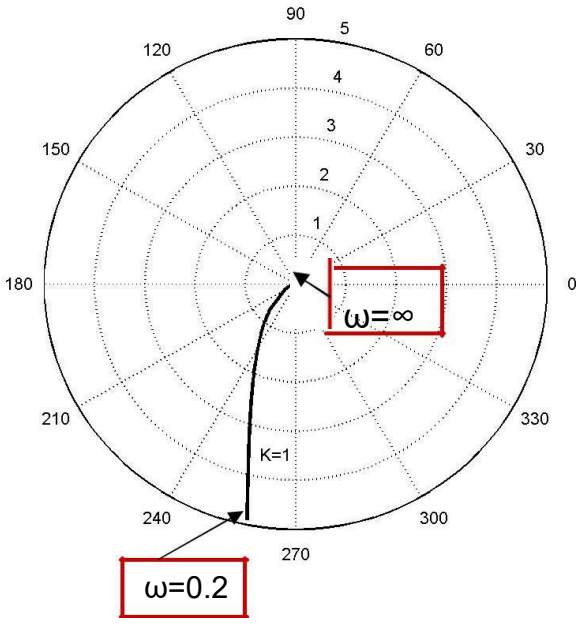
$$270^{0} \quad G(j\ )H(j\ )\ 180^{0} \tag{11.87}$$

Magnitude and phaseof the open-loopfrequency transfer function

No.	Frequency, rad/s	Magnitude	Phase, degrees
1	0	∞	270

2	**0.2**	**4.9029**	**259**
3	**0.4**	**2.3212**	**248**
4	**0.8**	**0.9761**	**231**
5	**1**	**0.7071**	**225**
6	**4**	**0.0606**	**194**
7	**10**	**0.01**	**186**
8	**50**	**0.0004**	**181**
9	**100**	**0.0001**	**181**
10	**200**	**≈0**	**≈180**

Polar plot of the transfer function $\frac{K}{s\,s\,1}$ and K=1



Example 11.7

Draw a polar plot of the open-loop transfer functionfor K=1, 10, 25, 55

$$GH \quad \frac{K}{s\,(\,s\,2)(\,s\quad 4)}$$

Solution

Frequency response

$$G\,(\,j\,)\,H\,(\,j\,)\quad \frac{K}{j\,\,(\,j\quad\quad 2)(\,j\quad\quad 4)}$$

Magnitude

$$|G(j\ )H(j\ )| \quad \frac{K}{^{2}\sqrt{4}\ \sqrt{^{2}\ 16}}$$

Angle

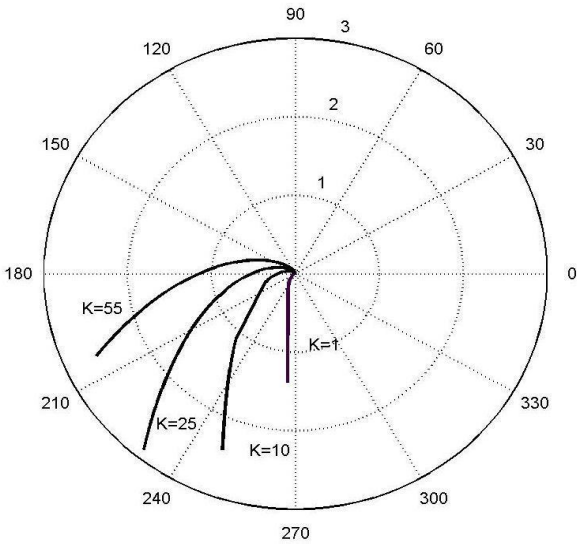
$$G(j\ )H(j\ ) \qquad \frac{}{2}\tan^{1} \quad \frac{}{2}\tan^{1} \quad \frac{}{4}$$

The lies in II and III quadrants as $90^{0}\ G(j\ )H(j\ )\ 270^{0}$

Magnitude and phase of the open-loop frequencytransfer function (K=1)

No.	Frequency, rad/s	Magnitude	Phase, degrees
1	0.1	1.2481	266
2	0.2	0.6211	261
4	0.4	0.3049	253
5	0.8	0.1423	237
6	1	0.1085	229
7	4	0.0099	162
8	10	0.0009	123
9	50	0	97

Polar plot of the transfer function $GH \quad \frac{K}{s(s\ 2)(s\ \ 4)}$ for K=1, 10, 25, 55



Example 11.8

Draw a polar plot of the open-loop transfer function $G(s)H(s) \quad \frac{K}{s^{2}(s1)}$

Solution

Frequency response

$$G(j\ )H(j\ ) \quad \frac{K}{(j\ )^2(j\ \ 1)}$$

Magnitude

$$\left|G(j\ )H(j\ )\right| \quad \frac{K}{{}^2\sqrt{{}^2\ \ 1}}$$

Angle

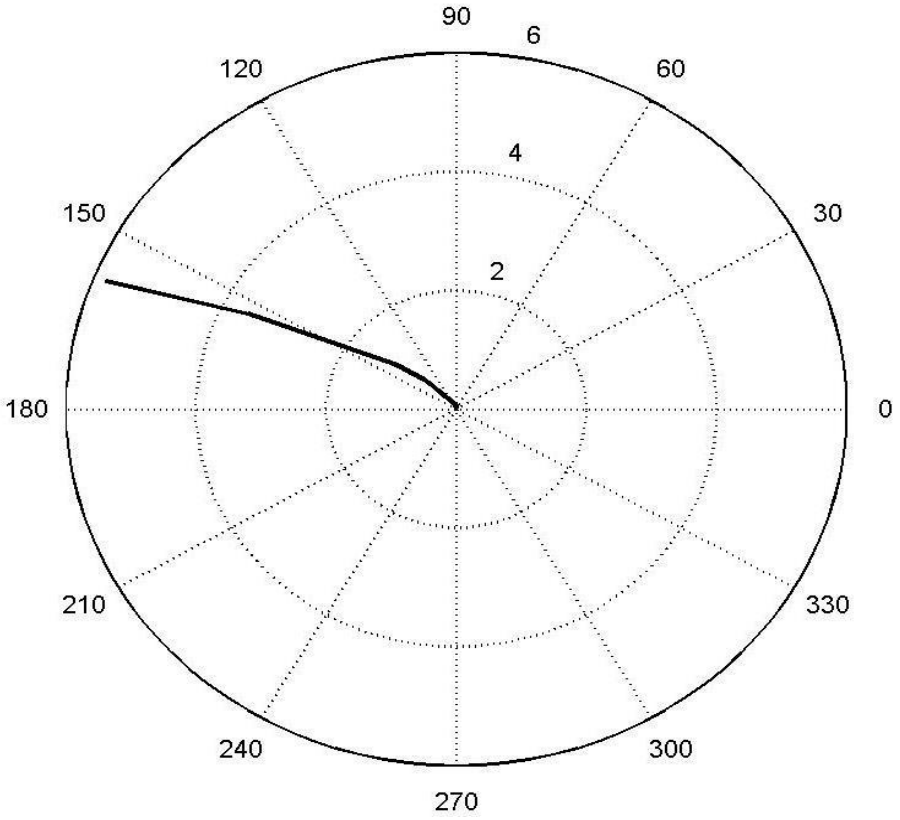
$$G(j\ )H(j\ ) \quad 180^{\ 0} \quad \tan^{\ 1}$$

The lies in II quadrant only as $90^{\ 0}\ G(j\ )H(j\ )\ 180^0$

Magnitude and phase of the open-loop frequency transfer function (K=1)

No.	Frequency, rad/s	Magnitude	Phase, degrees
1	0.4	1.10803	158
2	0.5	3.5777	153
4	0.8	1.2201	141
5	1	0.7071	135
6	2	0.1118	117
7	3	0.0351	108
8	4	0.0152	104
9	5	0.0078	101

Polar plot of the transfer function $GH \quad \frac{K}{s(s\ 2)(s\ \ 4)}$ for K=1, 10, 25, 55



Equation Chapter (Next) Section 1

Bode plot using MATLAB

Program 1: Sketch the bode plot for the open loop transfer function $\frac{C(s)}{R(s)} = \frac{40}{s(s+1)(s+4)}$. Determine the gain margin, phase margin, gain cross over frequency and phase cross over frequency.

Solution:

```
>> num=[0
40] num=
0        40
>> q1=[1 0];
>> q2=[1 1];
>> q3=[1 4];
>> den=conv(q1,q2);
>> den=conv(den,q3);
den=
1        9 24 16 0
>>sys=tf(num,den)
Transfer
       function: 1
--------------------
   s^4+9s^3+24s^2+16s
>>bode(sys)
>>margin(sys)
```

14. Nyquist plot

11.1.11.

Definition

Nyquist criterion is a graphical method of determining stability of feedback control systems by using the Nyquist plot of their open-loop transfer functions.

11.1.12. Theory

Feedback transfer function

$$\frac{C(s)}{R(s)} \quad \frac{G(s)}{1 \; G(s)H(s)} \tag{11.6}$$

Poles and zeros of theopen-loop transfer function

$$G(s)H(s) \frac{K(s \; z_1)(s \; z_2)...(s \; z_m)}{(s \; p_1)(s \; p_2)...(s \; p_n)} \tag{11.7}$$

$$1 \; G(s)H(s) \quad \frac{(s \; p_1)(s \; p_2)...(s \; p_n) \; K(s \; z_1)(s \; z_2)...(s \; z_m)}{(s \; p_1)(s \; p_2)...(s \; p_n)} \tag{11.8}$$

Number of closed-loop poles - Number of zeros of 1+GH = N umber of open-loop poles

$$1 \; G(s)H(s) \frac{(s \; z_{c_1})(s \; z_{c_2})(s \; z_{c_n})}{(s \; p_1)(s \; p_2)...(s \; p_n)} \tag{14.4}$$

$z_{c_1}, z_{c_2} ... z_{c_n}$ = zeros of 1+G(s)H(s)

These are also poles of the close-loop transfer function

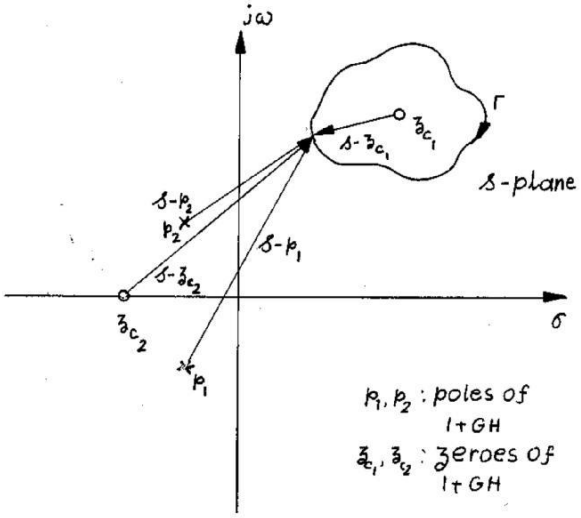
Magnitude

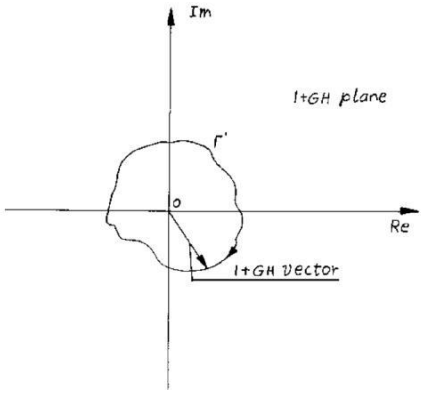
$$|1 \; G(s)H(s)| \quad \frac{|s \; z_{c_1}||s \; z_{c_2}|...|s \; z_{c_n}|}{|(s \; p_1)||(s \; p_2)|...|(s \; p_n)|.} \tag{14.5}$$

Angle

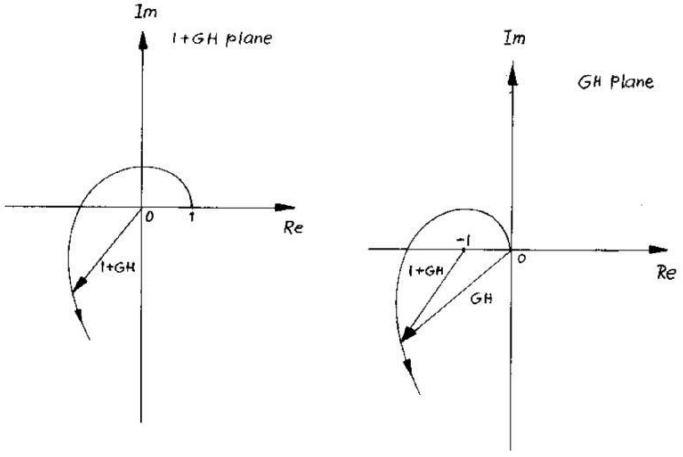
$$1 \; G(s)H(s) \frac{s \; z_c \quad s_{1} z_c \quad s_{2} z_c \quad _{n}}{(s \quad p_1)(s \quad p_2)(s \quad p_n)} \tag{14.6}$$

The s-plane to 1+GH plane mappingphase angle of the 1+G(s)H(s) vector, corresponding to a point on the s-plane is the difference between the sum of the phase of all vectors drawn from zeros of 1+GH(close loop poles) and open loops on the s plane. If this point s is moved along a closed contour enclosing any or all of theabove zeros and poles, only the phase of the vector of each of the enclosed zeros or open-loop poles will change by 360^0. The directionwill be in the same sense of the contour enclosing zeros and in the opposite sense for the contour enclosing open-loop poles.

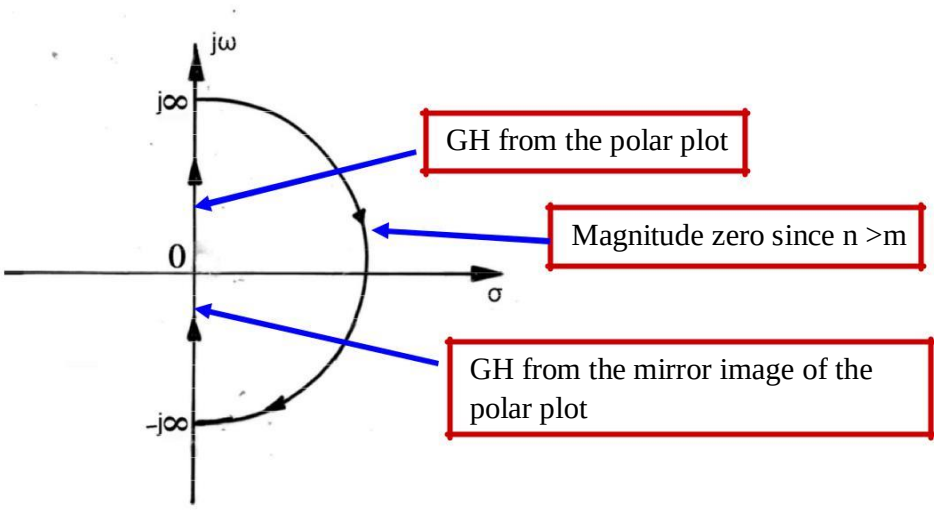




11.2. Principle of argument

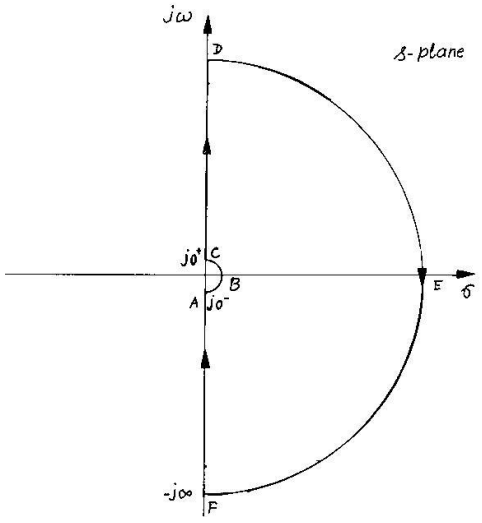


When a closed contour in the s-plane encloses a certain number of poles and zeros of 1+G(s)H(s) in the clockwise direction, the number ofencirclements of the origin by the corresponding contour in the G(s)H(s)plane will encircle the point (-1,0) a number of times given by thedifference between the number of its zeros and poles of 1+G(s)H(s) it enclosed on the s-plane.



Modified contour on the s-plane forchecking the existence of closed-looppoles

$s\ e^{j}$



Magnitude of GH remains the same alongthe contourPhase of β changes from 270 to 90 degrees

14.4. Gain Margin and Phase Margin

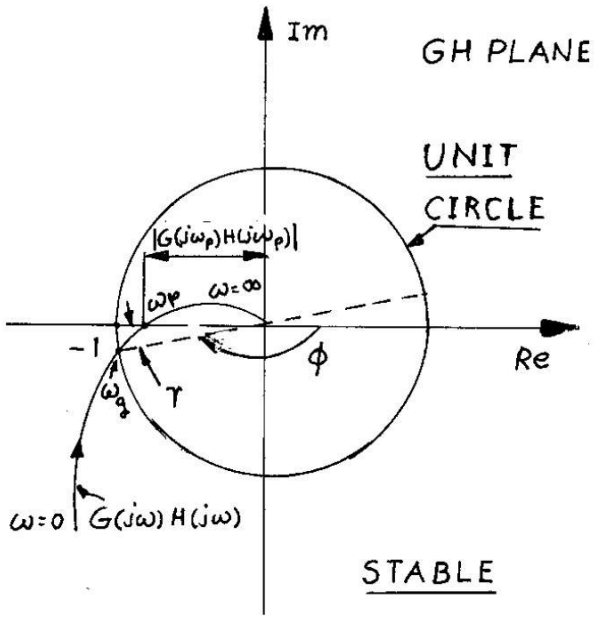
Phase crossover frequency $_p$ is the frequency at which the open-loop transfer function has a phase of 180^0. The gain crossover frequency $_g$ is the frequency at whichthe open-loop transfer function has a unit gain

Gain margin

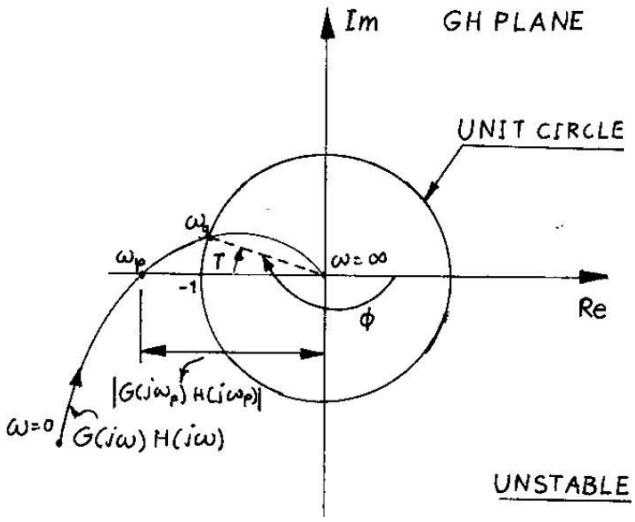
$$M \quad 20\log \quad \left|G(j\ _p)H(j\ _p)\right| \tag{14.7}$$

Phase margin

$$G(j\ _g)H(j\ _g)\ \ 180^o \tag{14.8}$$



(b)



14.1.10 Procedure

(1) Locate open-loop poles on the s-plane
(2) Draw the closed contour and avoid open-loop poles on the imaginary axis
(3) Count the number of open-loop poles enclosed in the above contour of step 2, say P
(4) Plot G(j)H(j) and its reflection on the GH plane and map part of the small semi-circle detour on the s-plane around poles (if any) on the imaginary axis.
(5) Once the entire s-plane contour is mapped on to the GH plane, count the number of encirclements of the point (-1,0) and its direction. Clockwise encirclement is considered positive, say N.
(6) The number of closed-loop poles in the right-half s-plane is given by Z=N+P. if Z >0, the system is unstable.
(7) Determine gain margin, phase margin, and critical value of open-loop gain.

Example 11.6

Using Nyquist criterion, determine the stability of a feedback systemwhose open-loop transfer function is given by

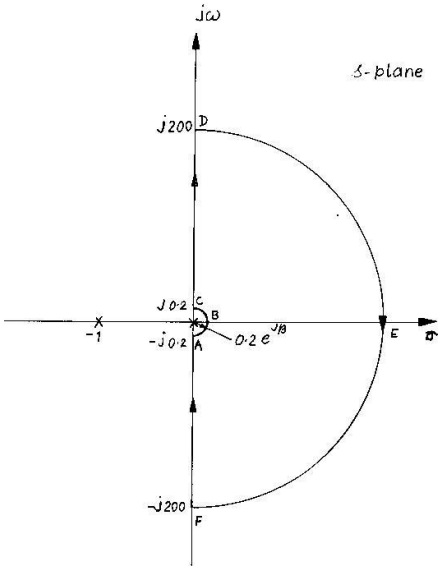
$$G(s)H(s) \quad \frac{K}{s(s\ 1)}$$

Solution

Step 1 Locate open-loop poles on the s-plane. Open-loop poles are at s=0 and –1. Let K=1

Step 2 Draw the closed contour on the s-plane to check the existenceof closed-loop poles in the right-half s-plane.

Open-loop poles and s-plane contour



$$\left| G(j\ )H(\right| \frac{1}{\sqrt{\quad}}$$

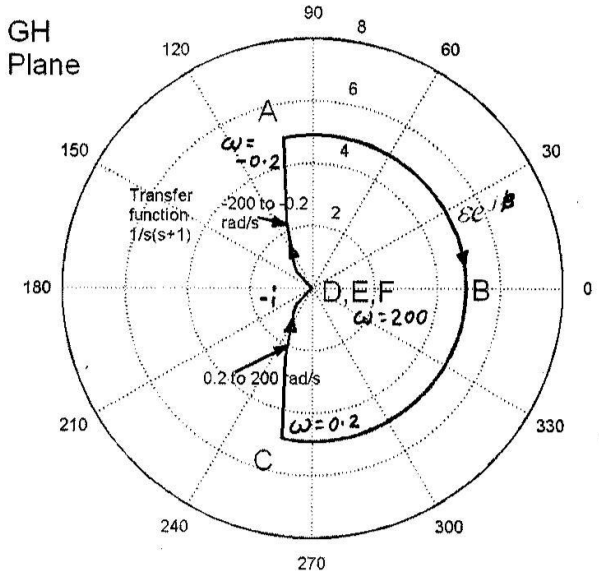
$$\sqrt{1} \qquad ^2$$

$$G(j\ )H(j\ ) \qquad \tan^{1}$$

$$\overline{2}$$

No.	Frequency, rad/s		Magnitude	Phase, degrees	, s-plane, deg	, GH plane, deg
1	0.2	Positive frequencies	4.9029	259	270	101
2	0.4		2.3212	248	280	91
3	0.8		0.9761	231	290	80
4	1		0.7071	225	300	69
5	4		0.0606	194	310	58
6	10		0.01	186	320	46
7	50		0.0004	181	330	35
8	100		0.0001	181	340	23

9	200		0	180	350	12
10	-200	Negative frequencies	0	180	0	0
11	-100		0.0001	179	10	348
12	-50		0.0004	179	20	337
13	-10		0.01	174	30	325
14	-4		0.0606	166	40	314
15	-1		0.7071	135	50	302
16	-0.8		0.9761	129	60	291
17	-0.4		2.3212	112	70	280
18	-0.2		4.9029	101	80	269



The above system is stable. Here, phase crossover frequency is very large (infinity) and gain crossover frequency 0.786 rad/s. Phase angle corresponding to gain crossover frequency= $232^0$and Phase margin is 52^o

Example 11.1.12.

Using Nyquist criterion, determine the stability of a feedback systemwhose open-loop transfer function is given by

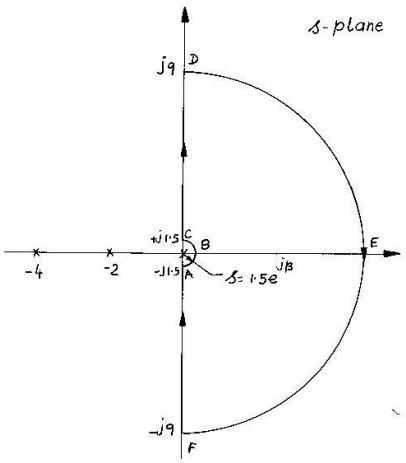
$$G(s)H(s)\ \frac{55}{s(s\ 2)(s\ \ 4)}$$

Solution

Step 1 Locate open-loop poles on the s-plane. Open-loop poles are at s=0, -2 and –4. Let K=1

Step 2 Draw the closed contour on the s-plane to check the existenceof closed-loop poles in the right-half s-plane.

Open-loop poles and s-plane contour



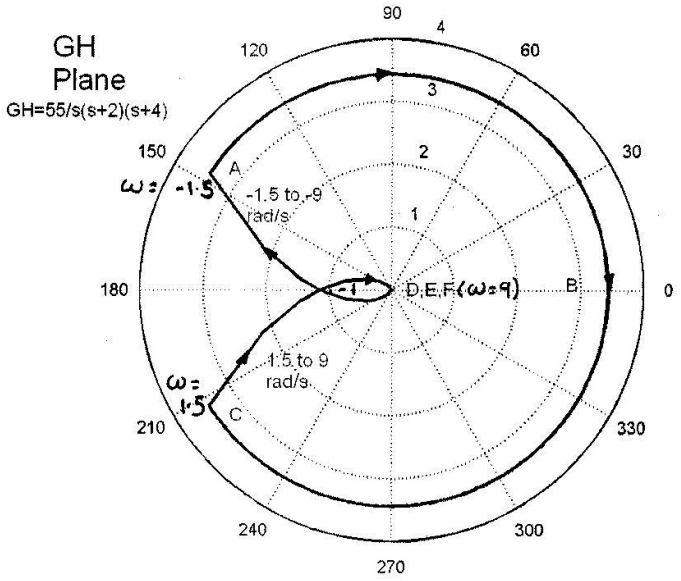
The number of open-loop pole enclosed, P is zero

$$|G(j\ )H(j\ )| \quad \frac{K}{\sqrt{\ ^2}\ 4\sqrt{^2}\ 16}$$

$$G(j\ )H(j\ ) \quad _\tan^{1} \quad _\tan^{1} \quad _$$
$$2 \qquad 2 \qquad 4$$

No.	Frequency		Magnitude	Phase, degrees	, s-plane, deg
1	1.5	Positive frequencies	3.4332	213	270
2	2		2.1741	198	280
3	2.5		1.4568	187	290
4	2.83		1.1446	180	300
5	3		1.017	177	310
6	3.5		0.7334	169	320
7	4.5		0.4122	156	330
8	5		0.319	150	340
9	1.105		0.2513	146	350
10	6		0.201	142	0
11	7		0.1339	136	10
12	8		0.0932	131	20
13	9		0.0673	126	30
14	-9	Negative frequencies	0.0673	234	40
15	-8		0.0932	229	50
16	-7		0.1339	224	60

17	-6		0.201	218	70
18	-1.105		0.2513	214	80
19	-5		0.319	210	90
20	-4.5		0.4122	204	0
21	-3.5		0.7334	191	343
22	-3		1.017	183	326
23	-2.83		1.1446	180	309
24	-2.5		1.4568	173	292
25	-2		2.1741	162	276
26	-1.5		3.4332	147	259



Here, Z=N+P=2.

Hence, the above system is unstable.

Again,

Phase crossover frequency 2.83 rad/s

The gain at which the system becomes marginally stable, $K^* = 55 / 1.1446 = 48$

Gain margin

$$M = 20 \log \left| G(j_p) H(j_p) \right|$$

$$= 20\log |1.1446| = 1.17 \text{ dB}$$

Gain crossover frequency =3 rad/s and the corresponding angle of GH=177^o

Phase margin=$177-180=-3^o$

Nyquist plot using MATLAB

Program 1: Sketch the nyquist plot for the open loop transfer function $\frac{C(s)}{R(s)} = \frac{40}{s(s+1)(s+4)}$.

Solution:

```
>> num=[040]
num=
0          40
>> q1=[1 0];
>> q2=[1 1];
>> q3=[1 4];
>> den=conv(q1,q2);
>> den=conv(den,q3);
den=
1          9 24 16 0
>>sys=tf(num,den)
Transfer
        function: 1
--------------------
   s^4+9s^3+24s^2+16s
>>nyquist(sys)
```

MODULE#4

Equation Chapter (Next) Section 1

11.10 CLOSED LOOP FREQUENCY RESPONSE

11.101. Peak Magnitude

$$M_r \quad 20\log \left|\frac{C(j\ )}{R(j\ )}\right| \text{dB} \tag{11.11.1}$$

3 dB is considered good

11.102. Constant M-circles for unity feedback systems

$$M(j\ ) \quad \frac{G(j\ )}{1 \quad G(j\ )} \tag{11.11.2}$$

$$G(j\ ) \quad x \quad jy \tag{11.11.3}$$

$$|M(j\ )| \quad \frac{\sqrt{x^2 \quad y^2}}{\sqrt{(1 \quad x)^2 \quad y^2}} \tag{11.11.4}$$

$$M^2(1 \quad x)^2 \qquad M^2 y^2 \quad x^2 \qquad y^2 \tag{11.11.5}$$

$$x^2(1 \qquad\qquad M^2) \qquad (1 \qquad M^2) \qquad\qquad y^2 \qquad\qquad 2M^2 x \quad M^2 \tag{11.11.6}$$

$$x^2 \quad y^2 \quad 2\frac{M^2}{1 \quad M^2} x \quad \frac{M^2}{1 \quad M^2} \tag{11.11.7}$$

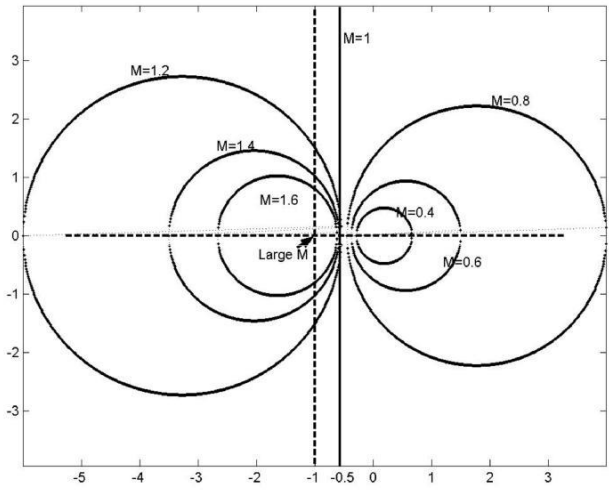
Adding $\frac{M^2 \quad 2}{1 \quad M^2}$ in both sides, we get

$$x \quad \frac{M^2 \quad 2}{1 \quad M^2} \quad y \quad ^2 \quad \frac{M}{1 \quad M^2} \quad ^2 \tag{11.11.8}$$

The above equation represents a family of circles with its center at $\frac{M^2}{1 \quad M^2}, 0$ and radius $\left|\frac{M}{1 \quad M^2}\right|$.

11.103. Family of M-circles

Family of M-circles corresponding to the closeloop magnitudes (M) of aunit feedback system Constant M-circles for unityfeedback systems



11.104. Constant N-circles

$$M \quad \frac{G(j)}{1 \; G(j)} \tag{11.11.9}$$

$$\tan^{1}\frac{y}{x} \tan^{1}\frac{y}{1 \; x} \tag{11.11.10}$$

$$N \tan\tan^{1}\frac{y}{x} \quad \tan^{1}\frac{y}{1 \; x} \tag{11.11.11}$$

$$\tan(A \; B) \quad \frac{\tan A \; \tan B}{1 \; \tan A \tan B} \tag{11.11.12}$$

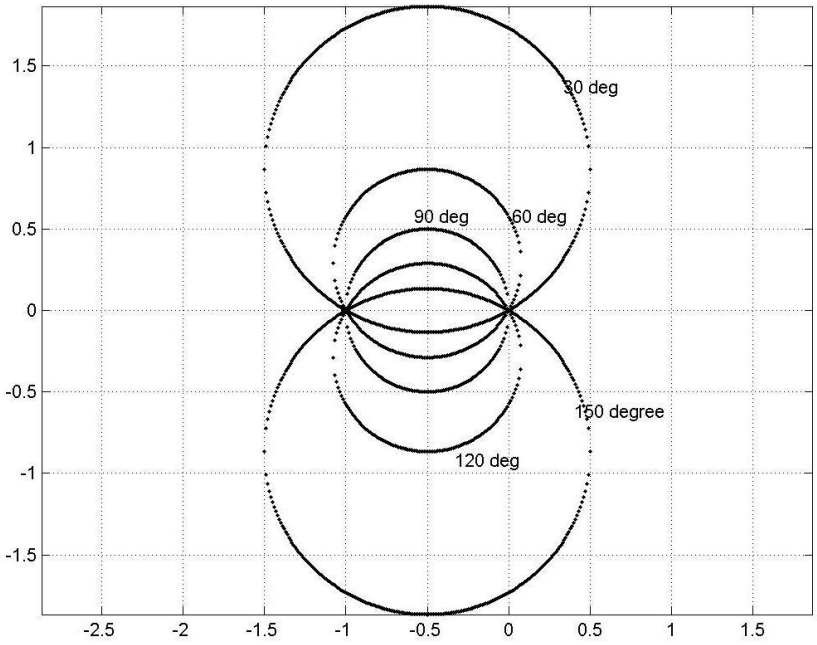
Here, tan( )=N

$$N \quad \frac{y}{x^{2} \; x \; y^{2}} \tag{11.11.13}$$

$$x \quad \frac{1^{2}}{2} \quad y \quad \frac{1}{2N}^{2} \quad \frac{1}{4} \quad \frac{1^{2}}{2N} \tag{11.11.14}$$

The above equation represents a family of circles with its center at ($\frac{1}{2}$, $\frac{1}{2N}$) and radius

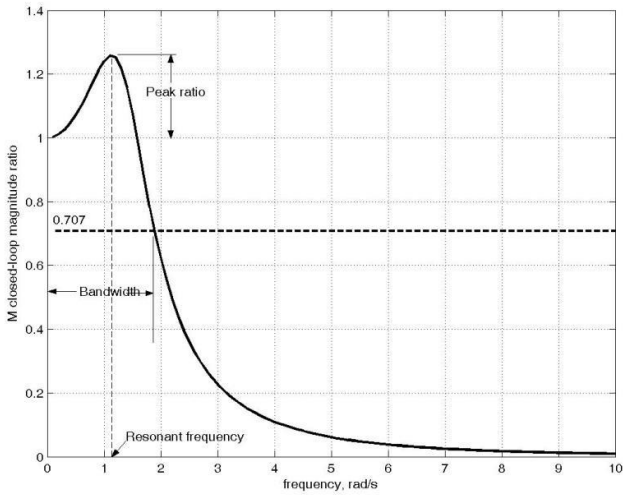
$$\sqrt{\frac{1}{4} \quad \frac{1}{2N}^{2}}$$

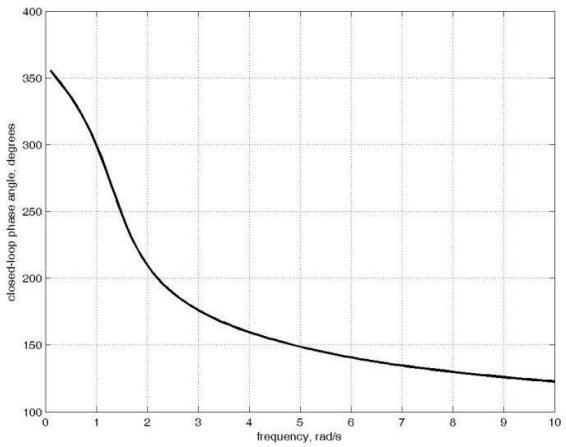


Example 11.101.

Determine the closed-loop magnitude ratio and bandwidth of the feedback system whose forward transfer function is given by $G(s) \quad \frac{10}{s(s\ 2)(s\ \ 4)}$ and H(s)=1, by (1) direct computation and (2) using M and N circles.

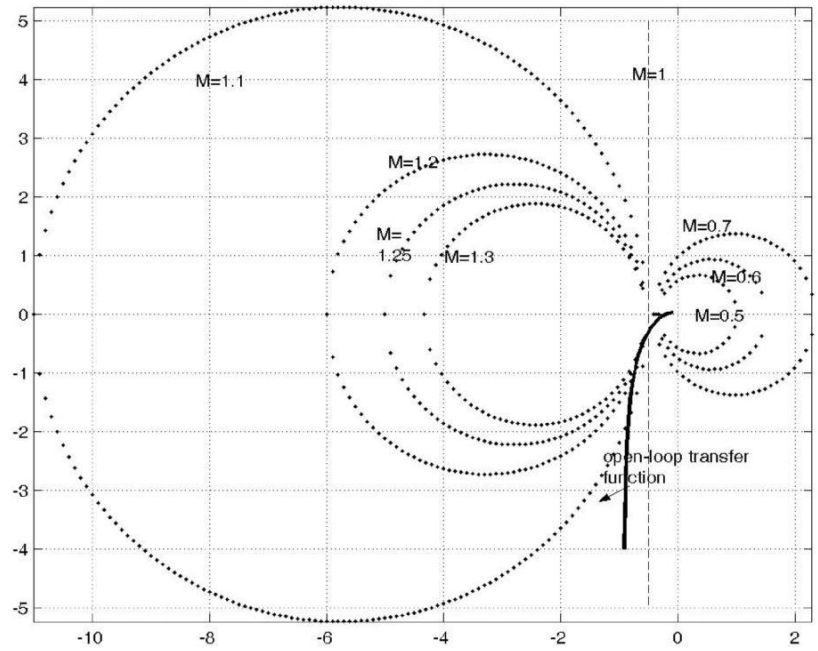
Solution

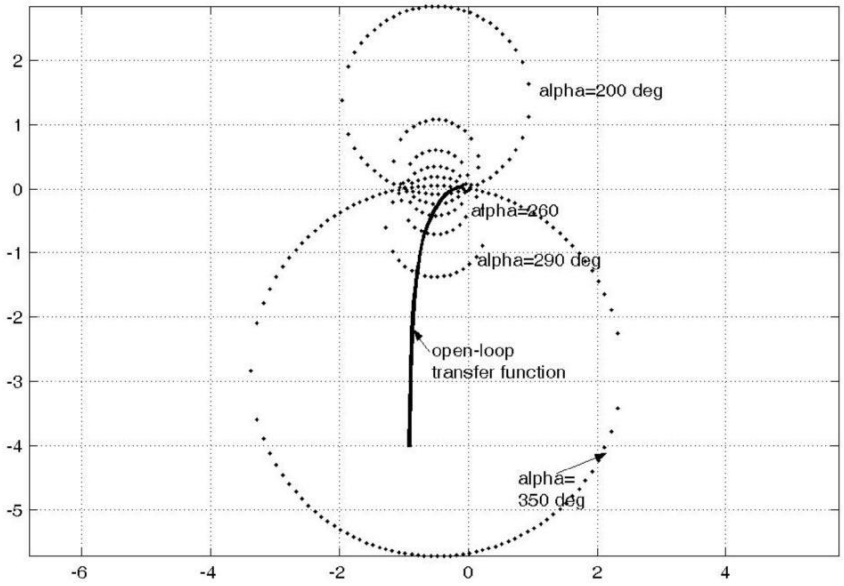




Closed-loopmagnitude and phase values

Frequency, rad/s	Closed-loop Magnitude ratio	Closed-loop Phase angle, deg
0.1	**1.0**	**355**
0.5	**1.1**	**335**
0.8	**1.2**	**316**
0.9	**1.2**	**308**
1.0	**1.2**	**300**
1.1	**1.3**	**290**
1.2	**1.3**	**280**
1.3	**1.2**	**269**
1.4	**1.2**	**258**
1.5	**1.1**	**248**
1.6	**1.0**	**238**
1.7	**0.9**	**230**
1.8	**0.8**	**222**
1.9	**0.7**	**216**
2.0	**0.6**	**210**





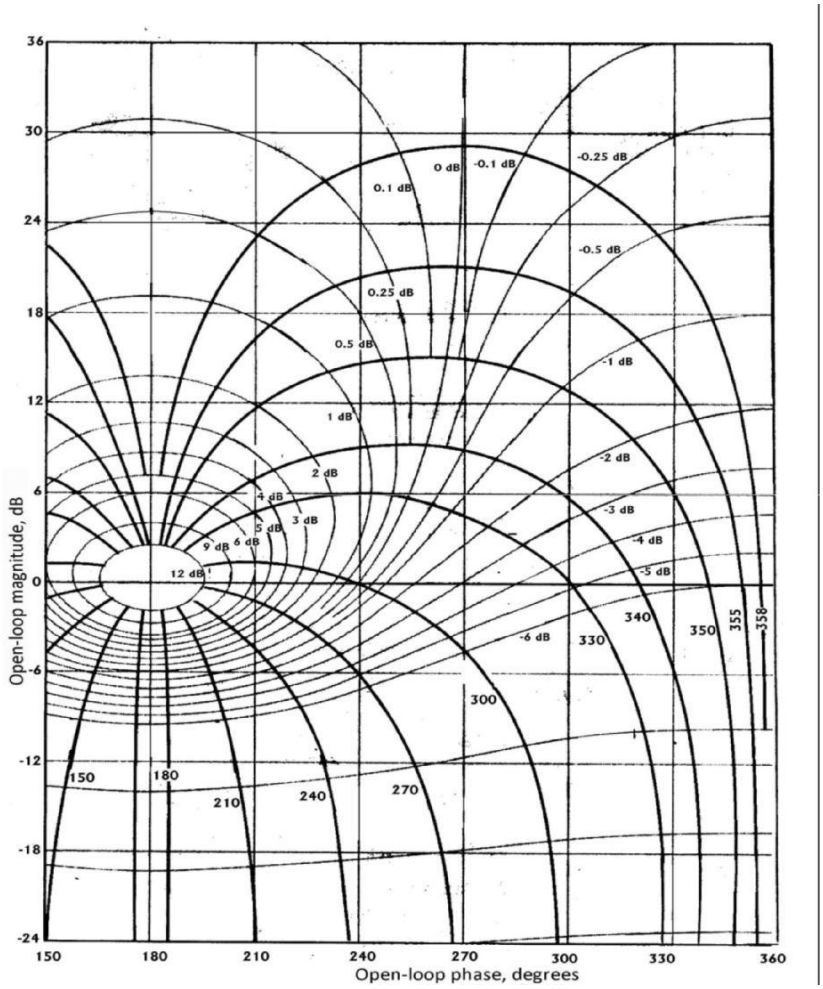
11.101.10 Nichols chart for closed-loop response

$$\frac{C(j\ )}{R(j\ )} \quad \frac{G(j\ )}{1\ G(j\ )H(j\ )} \tag{11.11.15}$$

$$M_{r} \quad 20\log \left|\frac{C(j\ )}{R(j\ )}\right| \text{dB} \tag{11.11.16}$$

$$\frac{C(j\ )}{R(j\ )} \tag{11.11.17}$$

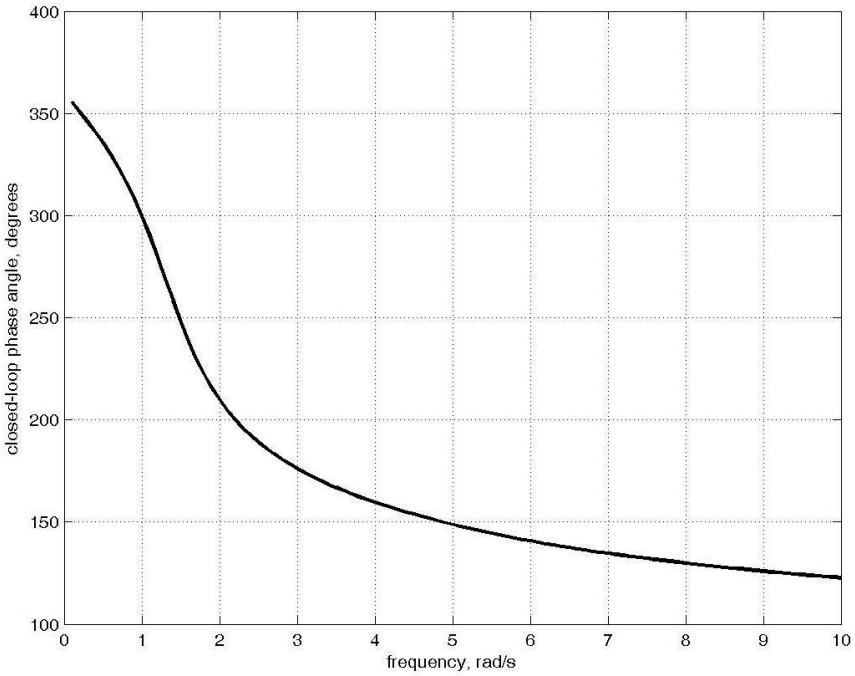
Closed-loopconstant magnitude (dB)and phase angle(degrees)lines of a unity feedbackfunction



Example 11.102.

Determine the closed-loop magnitude ratio and bandwidth of the feedback system whose forward transfer function is given by $G(s)\ \dfrac{10}{s(s+2)(s+4)}$ and H(s)=1, by (1) direct computation and (2) using Nichols chart.

Solution



Closed-loop magnitude and phase values

Frequency, rad/s	Closed-loop Magnitude ratio, dB	Closed-loop Phase angle, deg
0.1	0.02	355
0.2	0.10	351
0.3	0.22	346
0.4	0.38	341
0.5	0.59	335
0.6	0.84	330
0.7	1.11	323
0.8	1.39	316
0.9	1.66	308
1.0	1.87	300
1.1	1.99	290
1.2	1.95	280

1.3	1.72	269
1.4	1.27	258
1.5	0.62	248
1.6	-0.19	238
1.7	-1.12	230
1.8	-2.11	222
1.9	-3.13	216
2.0	-1.65	210

Open-loop magnitude and phase values

Frequency, rad/s	**open-loop magnitude, dB**	**open-loop phase angle, deg**
0.1	22	266
0.2	16	261
0.3	12	257
0.4	10	253
0.5	8	249
0.6	6	245
0.7	4	241
0.8	3	237
0.9	2	233
1.0	1	229
1.1	0	226
1.2	-1	222
1.3	-2	219
1.4	-3	216
1.5	-4	213
1.6	-5	210

1.7	-6	207
1.8	-7	204
1.9	-7	201
2.0	-8	198
2.1	-9	196
2.2	-10	193
2.3	-10	191
2.4	-11	189

Equation Chapter (Next) Section 1

11.11. Controllers

11.11.1. Basic Control Action and response of Control systems

An automatic controller compares the actual value of the plant output with the reference input (desired value), determines the deviation, and produces a control signal that will reduce the deviation to zero or to a small value. The manner in which the automatic controller produces the control signal is called the control action. Fig.1 is a block diagram of an industrial control system, which consists of an automatic controller, an actuator, a plant and a sensor (measuring element). The controller detects the actuating error signal, which is usually at a low power level, and amplifies it to a sufficiently high level. The output of the controller is fed to an actuator such as pneumatic motor or valve, hydraulic motor or electric motor. The actuator is the device that produces the input to the plant according to the control signal so that the output signal will approach the reference input signal.

The sensor or measuring element is device that converts the output variable into another suitable variable such as a displacement, pressure or voltage that can be used to compare the output to the reference input signal. This element is in the feedback path of the closed-loop system. The set point of the controller must be converted to a reference input with the same units as feedback signal from sensor.

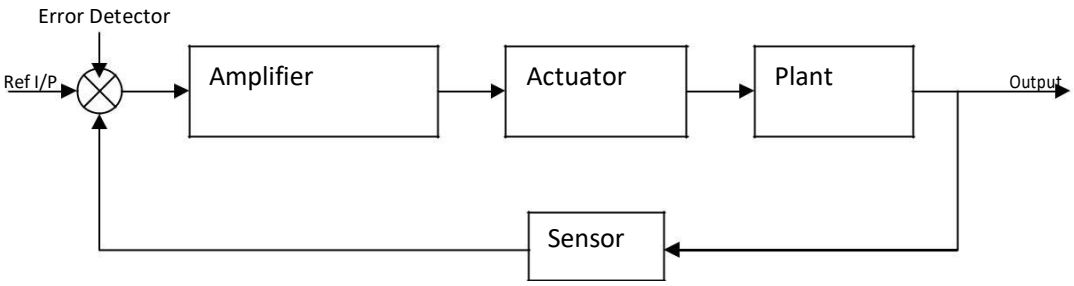


Fig.11.11.1. Basic Control Action and response of Control systems

11.11.2. APPLICATION OF CONTROL THEORY TO NON-ENGINEERING FIELDS

Engineering is concerned with understanding and controlling the materials and forces of nature for the benefit of human kind. Control system engineers are concerned with understanding and controlling segments of their environment often termed ***SYSTEMS*** to provide useful economic products for society. There has been considerable interest recently in applying the feedback control concepts to processes prevalent in the social, economic and political spheres. Some of the examples with block diagram models are discussed.

11.11.3. ECONOMIC INFLATION PROBLEM

A model of the vicious price-wage inflationary cycle, assuming simple relationship between wages, product costs and cost living is shown in Fig.1. The economic system depicted in this figure is found to be a positive feedback system.

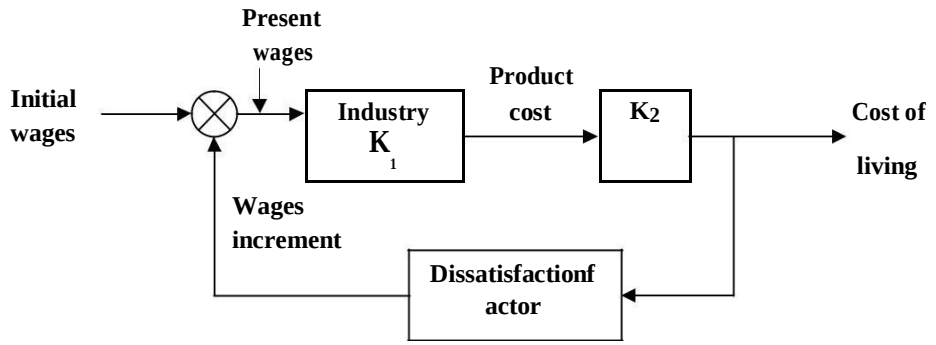


Fig.11.11.2. ECONOMIC INFLATION DYNAMICS

11.11.4. POLLUTION CONTROL IN AUTO ENGINE

To meet the emission standards for automobiles, Hydrocarbons (HC), Carbon monoxide (CO), and Nitrogen oxides (NO_x) emissions can be controlled by employing a three way catalyst in conjunction with a closed loop engine control system as shown in Fig.2. The exhaust gas sensor gives an indication of a rich or lean exhaust and compares it to a reference. The difference signal is processed by the controller, and the output of the controller modulates the vacuum level in the carburetor to achieve the best air-fuel ratio for proper operation of the catalytic converter.

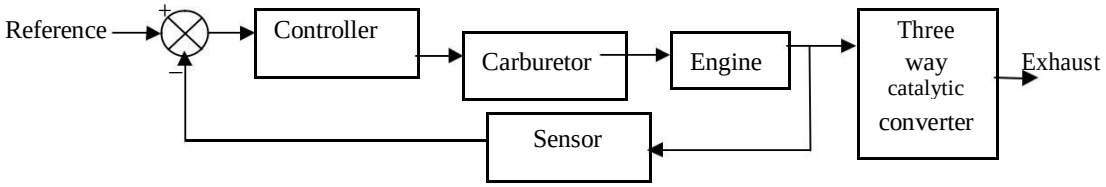


Fig.11.11.3

11.11.1.10 CONTROL OF BLOOD PRESSURE WITH ANESTHESIA

Anesthesia is used in surgery to induce unconsciousness. One problem with drug-induced unconsciousness is large differences in patient responsiveness. Furthermore the patient response changes during an operation. A model of drug induced Anesthesia Control is shown in Fig.3. The proxy for unconsciousness is arterial blood pressure.

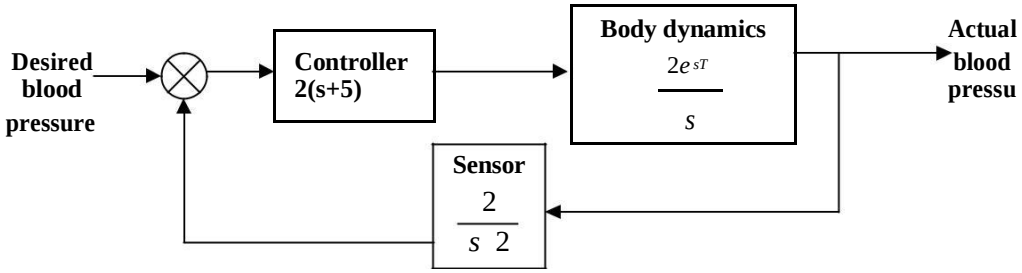
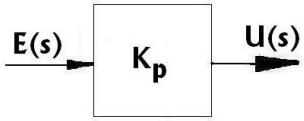


Fig.11.11.4

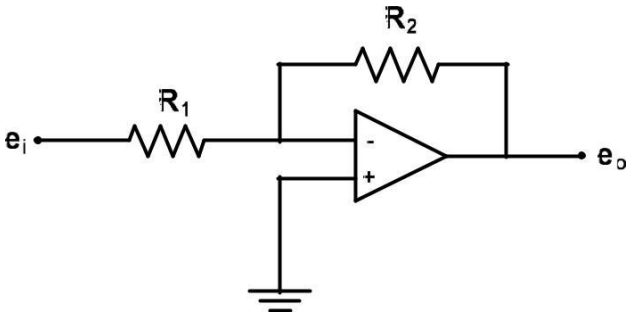
11.11.1.11. Types of Controllers

(i) P-controller
(ii) PI-controller
(iii) PD-controller
(iv) PID-controller

P-controller



(a)



(b)

Fig.11.11.5

Control system with P-controller with inertia load

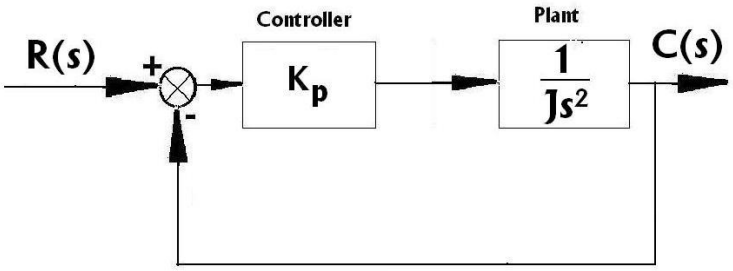


Fig.11.11.6

For this system, closed-loop response is

$$\frac{C(s)}{R(s)} \quad \frac{\frac{K_p}{Js^2}}{1 \quad \frac{K_p}{Js^2}} \quad \frac{K_p}{Js^2 \; K_p} \quad \frac{K_p}{J(s^2 \quad {}_n^2)} \tag{11.11.1}$$

$$C(s) \quad \frac{K_p}{J(s^2 \quad {}_n^2)\, R(s)} \tag{11.11.2}$$

For step-input, $R\,s \quad \frac{1}{s}$

Step response becomes

$$c(t) \qquad 1 \;\cos\; {}_n t \tag{11.11.3}$$

Where, ${}_n \quad \sqrt{\frac{K_p}{J}}$

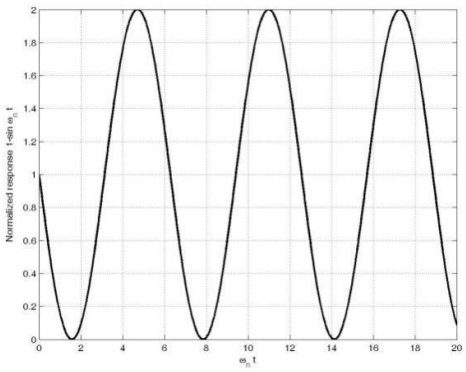


Fig.11.11.7

Solved problem

1. Consider the unity feedback system of Fig. 11.11.2. Let K_p=20 and J=50. Determine the equation of response for a unit step input and determine the steady-state error.

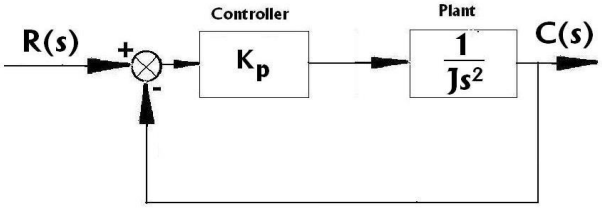


Fig.11.11.8

Solution

$$\frac{C(s)}{R(s)} \quad \frac{\frac{K_P}{Js^2}}{1 \quad \frac{K_p}{Js^2}} \quad \frac{K_p}{Js^2 \quad K_p}$$

$$\frac{C(s)}{R(s)} \quad \frac{K_p}{Js^2 \qquad {}_n^2}$$

$${}_n \quad \sqrt{\frac{K_p}{J}} \quad \sqrt{\frac{2}{5}} \text{ rad/s}$$

$$c(t) \frac{K_p}{J \; {}_n^2}(1 \;\cos\; t) \qquad {}_n$$

$$c(t) \quad 1 \;\cos\; \sqrt{\frac{2}{5}} \quad t$$

$$s_e(t) \quad 1 \quad 1 \;\cos\; \sqrt{\frac{2}{5}t} \quad \cos \quad \sqrt{\frac{2}{5}t}$$

2. Find the step response of the system shown inFig.11.11.9.

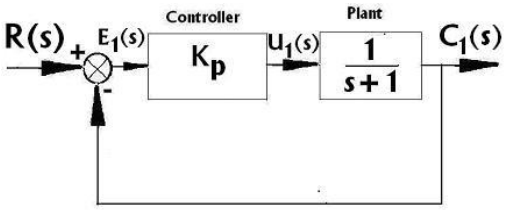


Fig.11.11.9

Solution

$$G_1 \quad \frac{K_p}{s \;\; 1}$$

$$\frac{E_1(s)}{R(s)} = \frac{1}{1 + G_1(s)} = \frac{s+1}{s+2}$$

$$E_1(s) = \frac{1}{1 + G_1(s)} R(s) = \frac{s+1}{s+2} R(s)$$

Step response is

$$E_1(s) = \frac{s+1}{s(s+2)}$$

$$\frac{U_1(s)}{R(s)} = \frac{K_p(s+1)}{(s+2)}$$

$$\frac{C_1(s)}{R(s)} = \frac{1}{s + 1 + K_p} = \frac{1}{s+2}$$

$$e_1(t) = \frac{1}{2}\left(1 + e^{-2t}\right)$$

$$c_1(t) = \frac{1}{2}\left(1 + e^{-2t}\right)$$

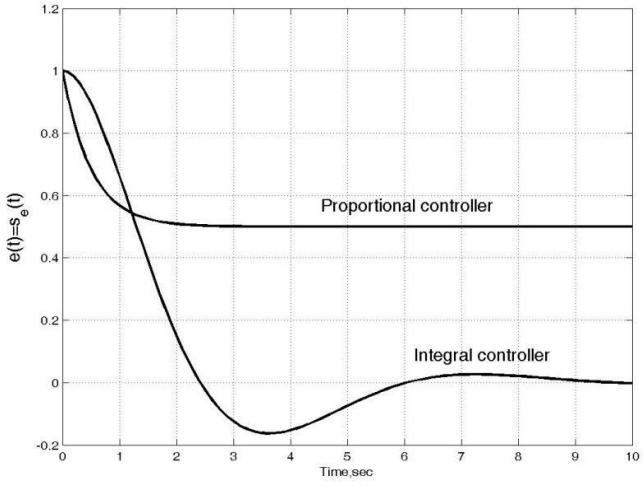
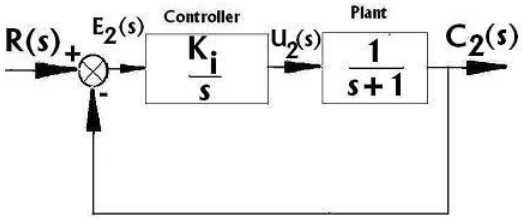
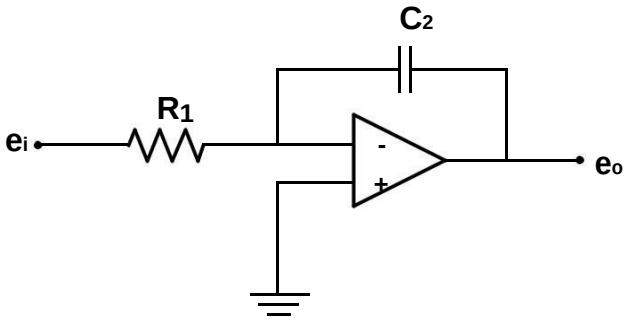


Fig.11.11.10

I-controller



(a)



(b)

Fig.11.11.11

$$G_2 \quad \frac{K_i}{s(s\ 1)}$$

$$\frac{E_2(s)}{R(s)} \quad \frac{1}{1\ G_2(s)} \quad \frac{1}{1\ \dfrac{K_i}{s(s\ 1)}} \quad \frac{s(s\ 1)}{s^2\ s\ 1}$$

$$\frac{U_2(s)}{R(s)} \quad \frac{K_i(s\ 1)}{s^2\ s\ 1}$$

$$\frac{C_2(s)}{R(s)} \quad \frac{1}{s^2\ s\ K_i} \quad \frac{1}{s^2\ s\ 1}$$

$$e_2(t)\ e^{\ 0.5t}\ \cos\ \frac{\sqrt{3}}{2}t\ \ \frac{1}{\sqrt{3}}\sin\ \frac{\sqrt{3}}{2}\ t$$

$$c_2(t)\ 1\ e^{\ 0.5t}\ \ \frac{1}{\sqrt{3}}\sin\ \frac{\sqrt{3}}{2}t\ \cos\ \ \frac{\sqrt{3}}{2}\ \ t$$

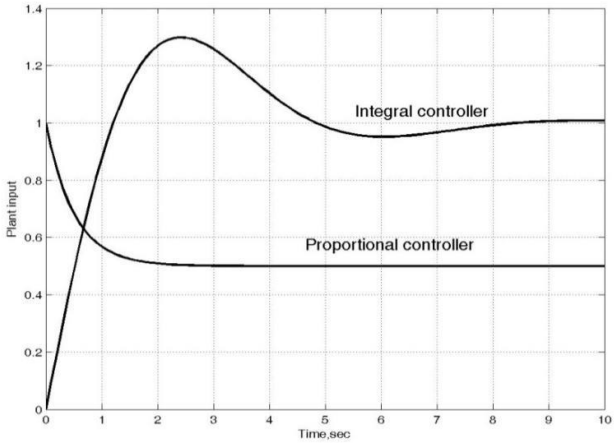
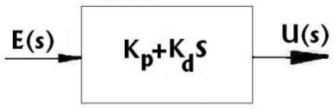
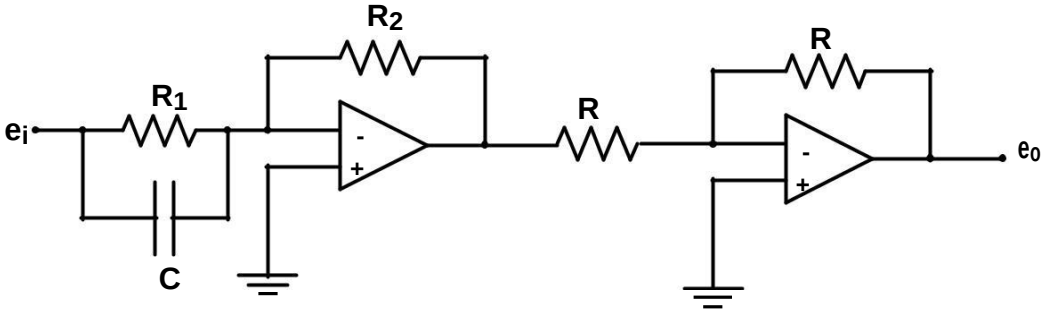


Fig.11.11.12

PD-controller



(a)



(b)

Fig.11.11.13

Control system with P-controller with inertia load

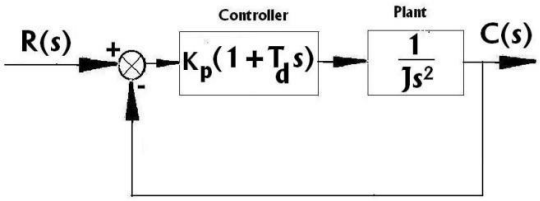


Fig.11.11.14

For this system, closed-loop response is

$$\frac{C(s)}{R(s)} \quad \frac{\frac{K_p(1 \ T_d s)}{Js^2}}{1 \ \frac{K_p(1 \ T_d s)}{Js^2}} \quad \frac{K_p(1 \ T_d s)}{Js^2 \ K_p(1 \ T_d s)} \quad \frac{K_p(1 \ T_d s)}{Js^2 \ \frac{K_p T_d}{J} s \ \frac{K_p}{J}} \tag{11.11.4}$$

$$C(s) \quad \frac{K_p(1 \ T_d s)}{Js^2 \ \frac{K_p T_d}{J} s \ \frac{K_p}{J}} R(s) \tag{11.11.5}$$

For step-input, $R(s) \quad \frac{1}{s}$

Step response becomes

$$c(t) \frac{K_p}{J_n^2} 1 e^{t} \quad \cos t_d \quad \frac{}{\sqrt{1 \ ^2}} \sin t_d t$$
$$\frac{K_p T_d}{J_n} \frac{1}{\sqrt{2}} e^{t} \sin \sqrt{1 \ ^2} t \tag{11.11.6}$$

Where, $_n \quad \sqrt{\frac{K_p}{J}}$

Solved problem

3. Consider the unity feedback system of Figure 3. Let K_p=20 and J=50. Determine the equation of response for a unit step input and determine the steady-state error. Here, K_p =20, T_d =1 and J=50.

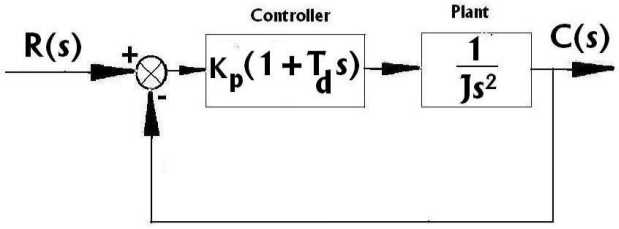


Fig.11.11
.14

Solution

$$\frac{C(s)}{R(s)} \quad \frac{20(s \ 1)}{50s^2 \ 20s \ 20} \quad \frac{20(1 \ s)}{50(s^2 \ 2 \ _n s \ _n^2)}$$

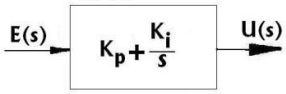
$$c(t) \quad \frac{K_p}{J_n^2}\left[1 - e^{-\zeta_n t}\left(\cos \omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t\right)\right]$$

$$\frac{K_p T_d}{J_n} \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$$

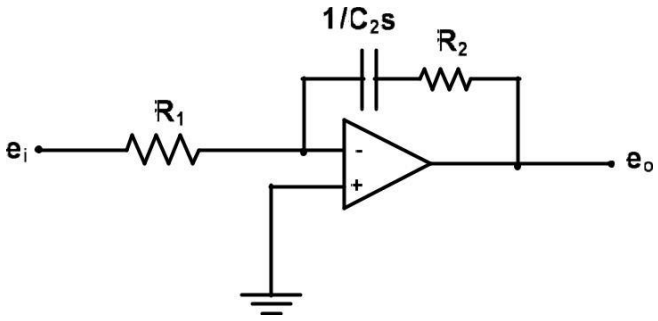
$$c(\infty) \quad \frac{K_p}{J_n^2}[1]$$

Transient characteristic	Only PD control No system damping
Maximum overshoot, %	31.1009 unsatisfactory
Rise time t_r, sec	3.15
Peak time t_p, sec	1.1024
Settling time t_s, sec (5% criterion)	15

PI-controller



(a)



(b)

Fig.11.11.15

Control system with PI-controller with inertia load

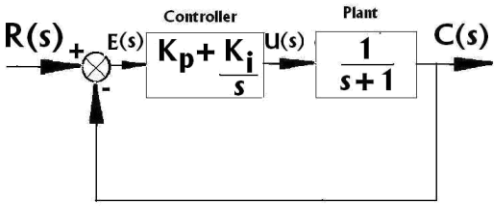


Fig.11.11.16

For this system, closed-loop response is

$$G \quad K_p \qquad \frac{K_i}{s} \qquad \frac{1}{s \;\; 1} \qquad \frac{sK_p \quad K_i}{s(s \;\; 1)} \tag{11.11.7}$$

$$\frac{E(s)}{R(s)} \;\; \frac{1}{1 \;\; G(s)_1} \qquad \frac{s(s \;\; 1)}{s^2 \;\; s(1 \;\; K_p) \;\; K_i} \tag{11.11.8}$$

$$\frac{U(s)}{R(s)} \qquad \frac{(s \;\; 1)(sK_p \quad K_i)}{s^2 \;\; s(1 \;\; K_p) \;\; K_i} \;\; 1 \tag{11.11.9}$$

$$\frac{C(s)}{R(s)} \;\; s^2 \; s(1 \;\; K_p) \;\; K_i \qquad \frac{(sK_p \quad K_i)}{(s \;\; 1)} \qquad \frac{1}{} \tag{11.11.10}$$

Step response

$u(t) \;\; 1$

$e(t) \;\; e^{\;t}$

$c(t) \quad 1 \;\; e^{\;t}$

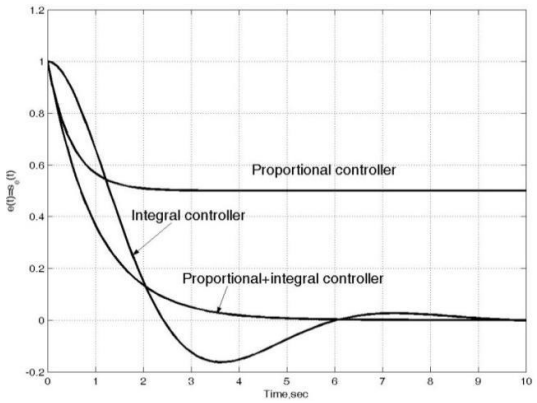


Fig.11.11.17

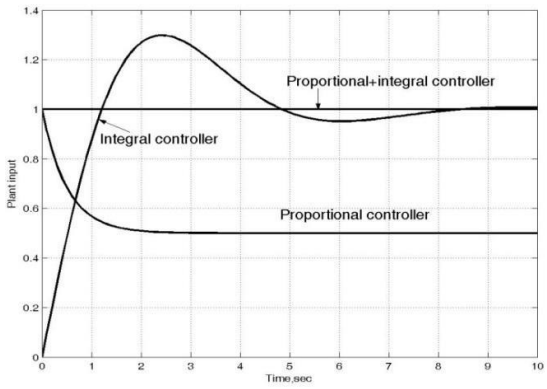


Fig.11.11.18

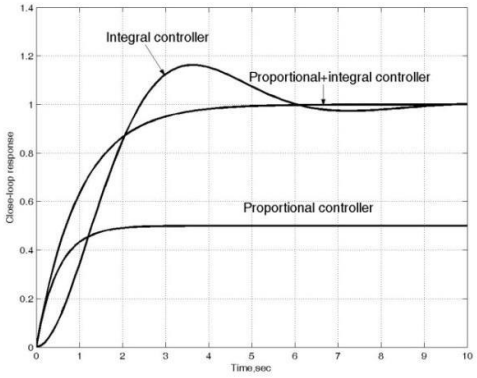
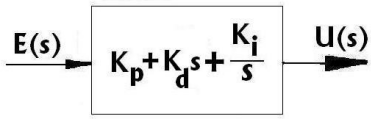
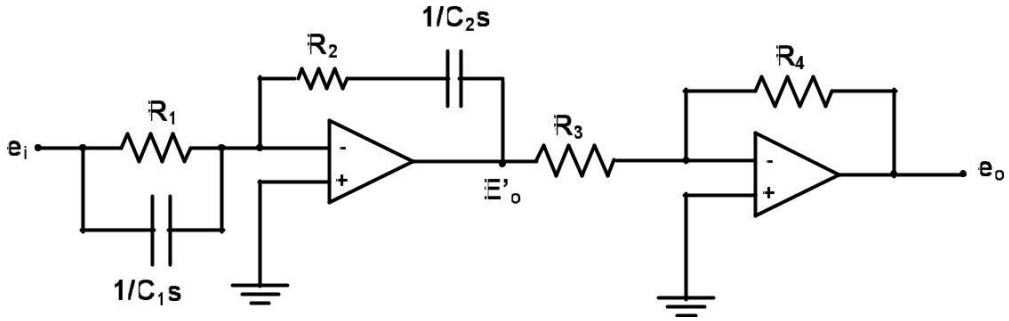


Fig.11.11.19

PID-controller



(a)



(b)

Fig.11.11.20

Here, transfer function of PID-controller,

$$G_c(s) = K_p + K_d s + \frac{K_i}{s} \tag{11.11.11}$$

$$G_c(s) = K_p\left(1 + T_d s + \frac{T_i}{s}\right) \tag{11.11.12}$$

Where,

$$T_i = \frac{K_i}{K_p}$$

$$T_d = \frac{K_d}{K_p} \tag{11.11.13}$$

Tuning of PID-controller

A. First Method (Ziegler and Nichols)

The Setup for obtaining system parameters for PID tuning

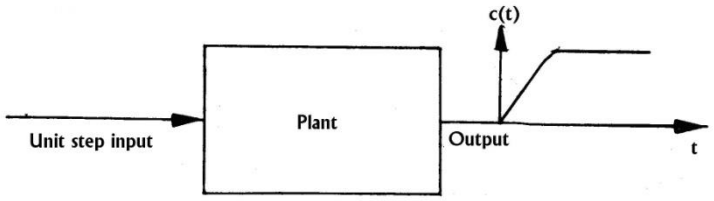


Fig.11.11.21

$$\frac{C(s)}{U(s)} = \frac{Ke^{-Ls}}{1 + Ts} \tag{11.11.14}$$

$$C(s) = \frac{Ke^{-Ls}}{1 + Ts} U(s) \tag{11.11.15}$$

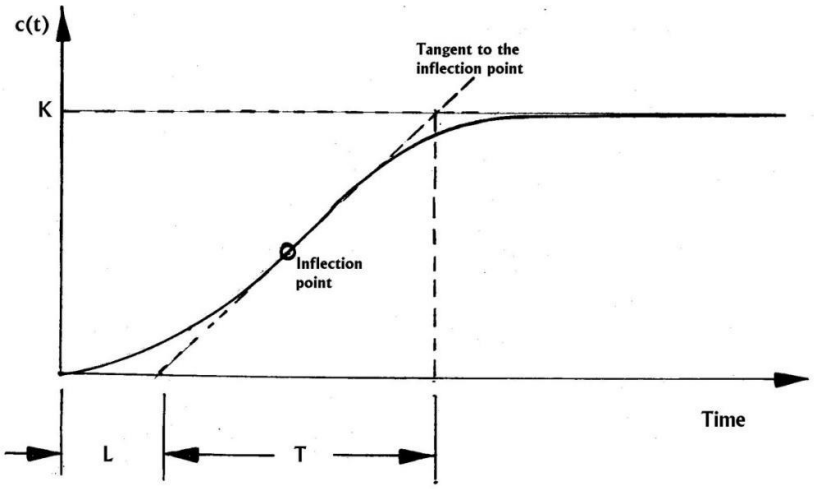


Fig.11.11.22

Ziegler-Nichols tuning rules based on step response

Type of controller	K_p	K_i	K_d
P	$\frac{T}{L}$	0	0
PI	$\frac{0.9T}{L}$	$\frac{L}{0.3}$	0
PID	$1.2\frac{T}{L}$	2L	0.5L

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

$$= \frac{1.2T}{L}\left(1 + \frac{1}{2Ls} + 0.5Ls\right)$$

$$G_c(s) = 0.6T\,\frac{\left(s + \frac{1}{L}\right)^2}{s}$$

Ziegler-Nichols tuning rule based on critical gain K_{cr} and critical period P_{cr}.

Type of controller	K_p	K_i	K_d
P	0.5 K_{cr}	0	0
PI	0.45 K_{cr}	1/1.2 P_{cr}	0
PID	0.6 K_{cr}	1/0.5P_{cr}	0.125 P_{cr}

Where, K_{cr} proportional constant of a switched-off integral and derivative controls at which sustained oscillations of period P_{cr} occur.

Second Method

$$G_c(s) = K_p\left(1 + \frac{T_i}{s}\right)\left(T_d s + \frac{1}{0.5P_{cr}s}\right) \qquad (11.11.16)$$

$$0.6K_{cr}\left(1 + \frac{1}{0.5P_{cr}s}\right)\left(0.125P_{cr}s\right)$$

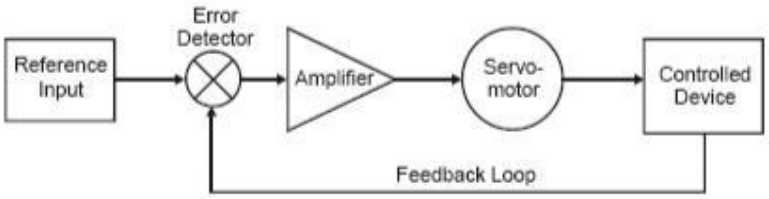
$$G_c(s) = 0.075K_{cr}P_{cr}\frac{s^2 + \left(\frac{4}{P_{cr}}\right)^2}{s} \qquad (11.11.17)$$

11.12. Components

11.12.1. AC SERVOMOTORS

A two phase servomotor (Induction Motor) (A few watts to hundred watts) is commonly used in feedback control systems. In servo applications, an induction motor is required to produce rapid accelerations from standstill.

Schematic Diagram

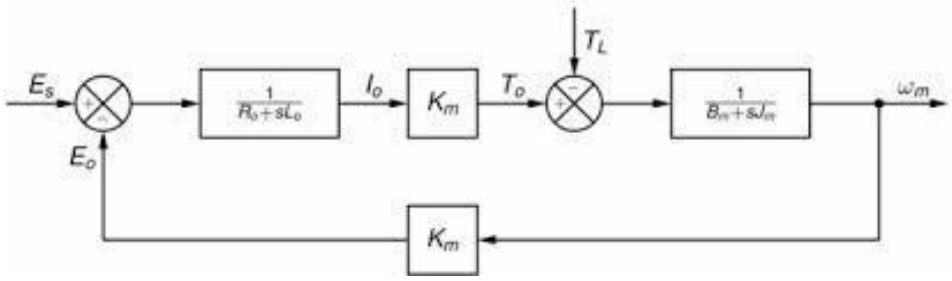


Constructional features

Squirrel Cage rotor with Cu or Al
conductor High Rotor resistance
Small diameter to length ratio to minimize inertia
Two stator windings in space quadrature(One called reference winding and the other Control winding)
The two voltages to stator windings must derived from same source(Or they must be in synchronism)

Principle of Operation

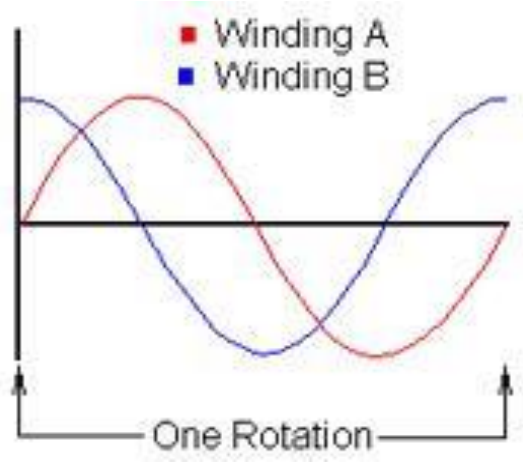
(i) The two applied AC voltage to stators with a phase difference produce a rotating flux.
(ii) As this moving flux sweeps over the rotor conductors, small emf is induced in rotor. Rotor being short circuited, currents will flow and this current interacts with rotating flux to produce a torque in the rotor. This torque causes the rotor to turn so that it chases the rotating magnetic flux.



Torque-Speed Characteristics of AC Servomotor

For induction motor in high power applications, rotor resistance is low in order to obtain maximum torque. Positive slope part of the characteristics is not suitable to control applications as this results instability.

In AC servomotor high resistance rotor results in negative torque-speed characteristics.This characteristic is needed for positive damping and good stability. The rotor has a small diameter –to-length ratio to minimize the moment of inertia and to give a good accelerating characteristic. However, more rotor resistance results more loss and less efficiency.



Transfer Function

The torque developed is a function of shaft angular position (Θ) and control voltage E_c.

$$G(s) \quad \frac{s}{E_c(s)} \quad \frac{K_m}{Js^2 + Ds} \quad \frac{K_m}{s(Js + D)} \quad \frac{K}{s(T_m s + 1)}$$

Where, $K = \frac{K_m}{D}$ = motor gain constant, $T_m = \frac{J}{D}$ = motor time constant

Merits of AC Servomotors

(i) Lower cost,
(ii) less weight and inertia,
(iii) higher efficiency and
(iv) fewer maintenance requirements(since no commutator or brush)

Demerits of AC Servomotors

(i) Nonlinear characteristics,
(ii) Used for low power applications(e.g. instrument servo),
(iii) Difficult for speed control and positioning

11.12.2. Synchros

It is also known as selsyn. It is a self-synchronizing device widely used in servomechanisms as a position indicator.

Important synchro systems are

Synchro system with transmitter and control transformer
Synchro system with synchro transmitter and motor
Synchro system with transmitter, differential and motor

General Constructional features of Synchro

(i) The construction of synchro transmiiter, motor and transformer are almost same.

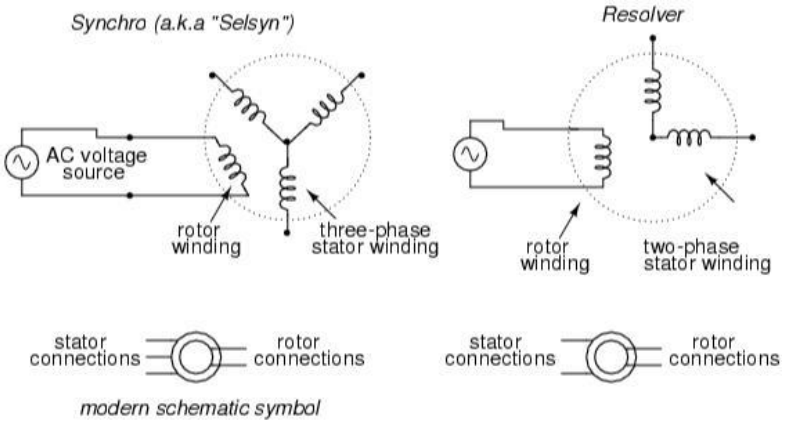
(ii) Stator laminated silicon steel, slotted to house distributed 3-ϕ,Y-connected windings with axes 120^0 apart.

(iii) Stator not directly connected to supply

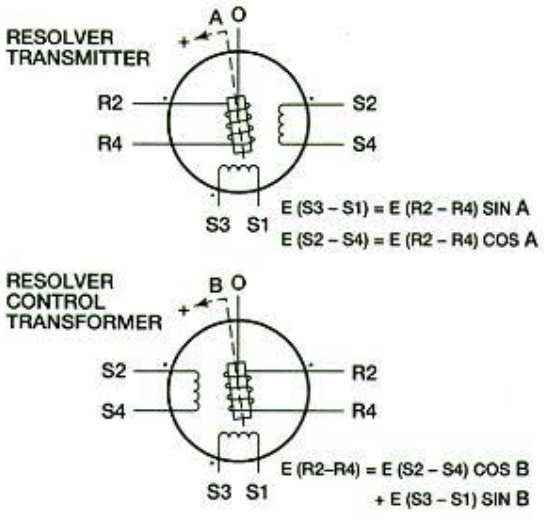
(iv) Rotor is 2-pole (dumb-bell shaped for synchro transmitter and cylindrical shape for control transformer) with single winding connected to AC source. The magnetic field in excited rotor induces voltages in stator coils. The magnitude of voltage induced in any stator coil depends on the angular position of coil's axis with respect to rotor axis.

(v) Synchro control transformer has cylindrical shape rotor so that air gap flux is uniformly distributed around the rotor.

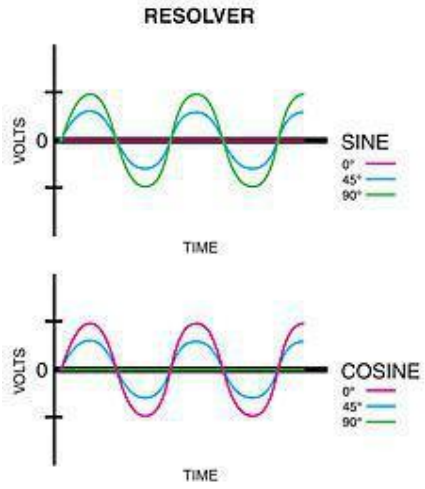
Constructional features



(a) Constructional features



(b) Electrical Circuit



(c) Schematic Symbol

Fig. Synchro Transmitter

Synchro transmitter

It is not a three phase machine. It is a single phase machine. Here, input is angular position of its rotor shaft.Output is a set of three stator coil-to-coil voltages. Common connection between the stator coils is not accessible.

Synchro system with transmitter and control transformer

A synchro error detector system may consist of synchro transmitter and synchro control transformer.

It compares two angular displacements and the output voltage is approximately linear with angular difference or misalignment between shafts of transmitter & Control transformer.

Usedas error detector in feedback systems.

Synchro system with synchro transmitter and synchro motor

The rotors of both the synchro devices are connected to same AC source. Figure (b) shows a circuit configuration, using two synchros, for maintaining synchronism between two shafts. When rotor windings are excited, emfs are induced by transformer action in the stator windings of transmitter and motor. If the two shafts are in similar positions (relative to that of the stator windings), then there are two emfs of equal value are induced in the two stator windings.Also no circulating current exists and hence no torque is produced. If the two shaft positions do not match, the emfs are unequal and result circulating current to flow. The circulating current in conjunction with air gap magnetic field produce torque which tend to align the shafts.

Synchro system with transmitter, differential and motor

The function of this system is to permit the rotation of a shaft to be a function of sum or difference of the rotations of two other shafts. The differential has 3-phase distributed windings on both stator and rotor. The voltages impressed on its stator windings induce corresponding voltages in its rotor windings.

$_r$=Displacement of receiver shaft

$_s$=Displacement of transmitter shaft

$_d$=Displacement of differential shaft

Then, $_r$ $_t$ $_d$ $_t$

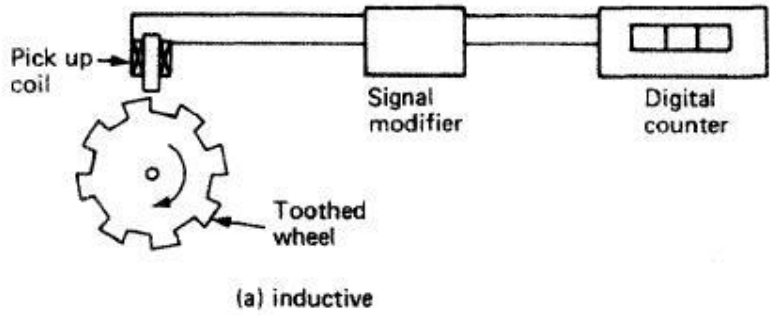
If the phase sequence of stator and rotor windings of differential are reversed then $_r$ $_t$ $_d$ t

11.12.3. TACHOGENERATOR OR TACHOMETER

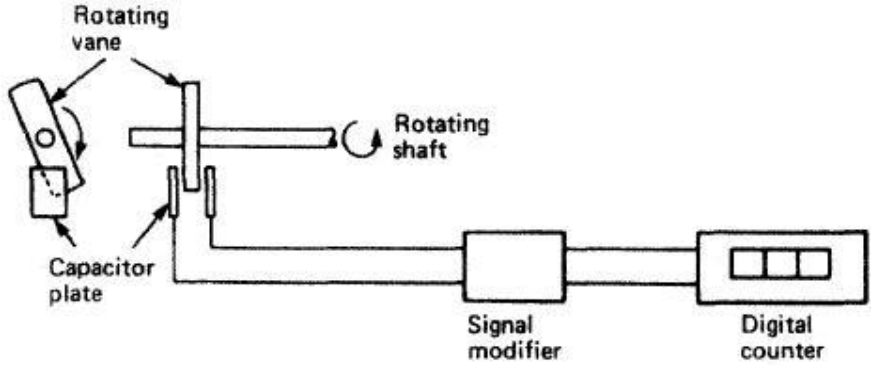
In many control systems, it is necessary to feedback a voltage proportional to speed of shaft. Tachogenerator serves the purpose.

11.12.3.1. DC Tachometer

It is a permanent magnet DC generator. It resembles a small DC machine having a PM stator, rotating armature, brush and commutator assembly. The rotor is connected to the shaft to be measured. The output voltage is proportional to the angular velocity of the shaft. The direction of rotation decides the polarity of the voltage.



(a) inductive



(b) Capacitive

DC tachogenerator suffers from the drawback of output in ripple, commutator & brush problem.

A DC tachometer can be used in AC servomechanism by converting the DC output voltage to an AC voltage by using a rectifier circuit.

11.12.3.2. AC Tachometer

Used in AC servomechanism. It resembles 2-phase AC induction motor.

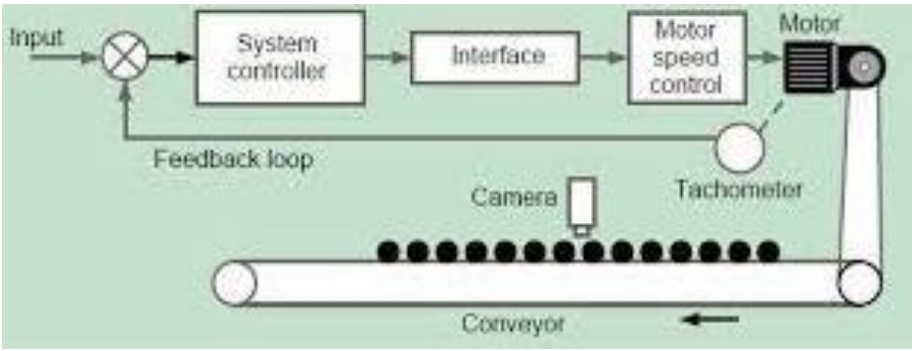
It comprises two stator windings arranged in space quadrature and a rotor which is not conductively connected to external circuit.

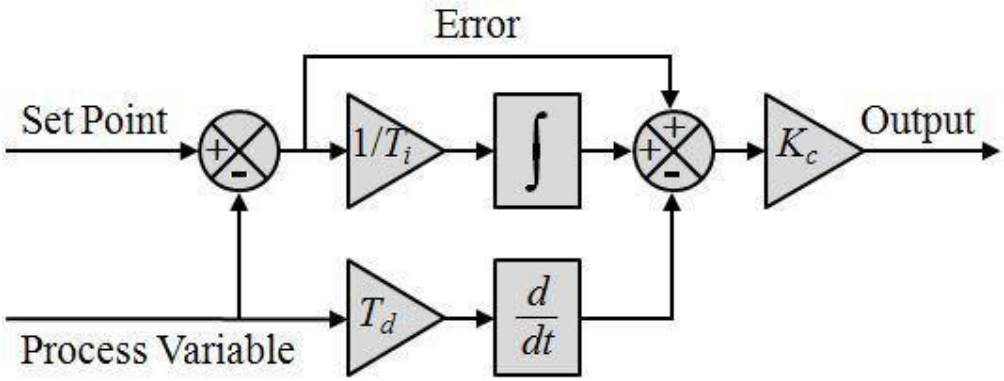
One stator phase winding is excited by a suitable AC voltage of constant magnitude and frequency. A voltage of the same frequency is generated across the other winding known as control winding.

It is necessary that the voltage developed across the control winding is linearly proportional to shaft speed and the phase of this voltage be fixed with respect to voltage applied to reference winding.

The output voltage is connected to high impedance circuit of amplifier so that the winding is considered open circuit.

An AC tacogenerator should have low inertia when rapid speed variations are encountered. The drag cup construction gives low inertia and is used many times





11.12.4. HYDRAULIC OPERATED DEVICES IN FEEDBACK CONTROL SYSTEM

Hydraulic systems must be stable and satisfactory under all operating conditions. Hydraulic output devices are generally of two types.

A. Hydraulic linear actuator(Produce linear motion)

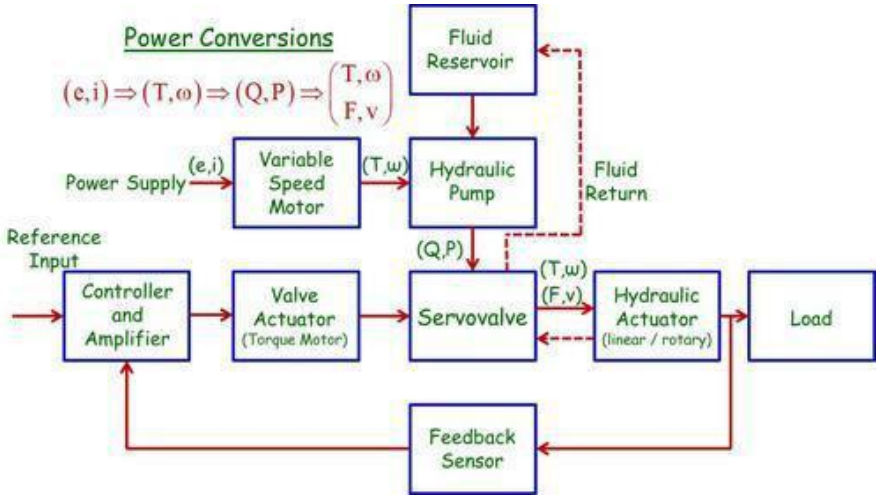
B. Hydraulic motors(rotary motion)

11.12.1.101 Hydraulic Linear Actuator(Hydraulic Servomotor)

Hydraulic linear actuator consists of pilot valve and a power cylinder. The piston inside the power cylinder divides the cylinder into two chambers. The pilot valve is known as spool valve because of its shape control the flow rate of the hydraulic fluid to the power cylinder. It is a four port valves. It is connected to fluid supply at constant pressure. The two ports connected to each chamber of power cylinder. One drain port is connected to reservoir.

Principle of Operation

If input x moves the pilot valve to the right, port II is uncovered, and so high pressure oil enters the right side of the power piston. Since port I is connected to the drain port, the oil in the left side of the power piston is returned to the drain. The oil flowing into the power cylinder is at high pressure; the oil flowing out from the power cylinder into the drain is at low pressure. The resulting difference in pressure on both sides of the power piston will cause it to move to the left.



Transfer Function

Rate of flow of fluid Q(kg/sec) time dt(sec) is equal to the power piston displacement dy(m) times the piston area A(sq.m) times the density of fluid P (kg/m^3).Fluid flow rate is proportional to pilot valve displacement x. So,

$Q \quad x$

$Q \; Kx$

where K is a positive constant.

$$Q \qquad \frac{dy}{} \qquad AP$$

$$AP \frac{dy}{dt} \; Kx$$

$APsY\ s \qquad KX\ s$

$$\frac{Y\ s}{X\ s} \quad \frac{K}{APs}$$

Advantages

(i) Hydraulic fluid acts as a lubricant and heat transfer medium
(ii) Comparatively small size hydraulic actuators can develop large forces or torques
(iii) Fast start, stop, and speed reversals(Faster response)
(iv) Hydraulic actuators can be operated under any type of load (i.e. continuous, intermittent, reversing or stalled loads)
(v) Availability of linear and rotary actuators(e.g. motors)
(vi) Better speed regulation.

Disadvantages

(i) Hydraulic fluid acts as a lubricant and heat transfer medium
(ii) Like electric power, hydraulic source not readily available
(iii) Presence of dirt contaminate the hydraulic fluid
(iv) Fire and explosion hazards exist
(v) For a similar function, cost of hydraulic system may be higher compared to electrical system.

11.12.1.102 DC and AC Motors in Control Systems to Position an Inertia Load

i. The **load** may be massive (e.g. radar antenna) or light weight precision instrument.
ii. The **actuator** should have
Desired dynamic response
Desired cost, size, and weight
iii. Electric power is readily available, cleaner and quieter and easier to transmit.
So electric motors is mostly preferable compared to hydraulic and pneumatic actuation)

Merits of DC motor

(i) Linear characteristics,
(ii) Used for large power applications,
(iii) Easier control

Demerits of DC motor

(i) Lower torque to volume and
(ii) Lower torque to inertia ratio.

Future developments

(i) Development of rare earth magnet results in DC motor high torque to volume ratio.
(ii) Advances in brush commutator technology make trouble free maintenance.
(iii) Development of brushless DC motors.

SUMMARY

CONTROL SYSTEM ENGINEERING

3.0 Introduction to Control system
- 3.1 Scope of Control System Engineer
- 3.2 Classification of Control System
- 3.3 Historical development of Control system
- 3.4 Analogus systems
- 3.5 Transfer function of Systems
- 3.6 Block diagram representation
- 3.7 Signal Flow Graph(SFG)

4.0 Feedback Characteristics of Control systems and sensitivity measures
- 1.6 The Concept of Feedback and Closed loop control
- 1.7 Merits of using Feedback control system
- 1.8 Regenerative Feedback

3.0Control System Components
- 3.1 Potentiometers
- 3.2 DC and AC Servomotors
- 3.3 Tachometers
- 3.4 Amplidyne
- 3.5 Hydralulic systems
- 3.6 Pneumatic systems
- 3.7 Stepper Motors

4.0 Time Domain Performance Analysis of Linear Control Systems
- 1.6 Standard Test Signals
- 1.7 Time response of 1^{st} order Systems
- 1.8 Unit step response of a prototype 2^{nd} order system
- 4.4 Unit Ramp response of a second order system
- 4.4 Performance Specification of Linear System in Time domain
- 4.5 The Steady State Errors and Error Constants
- 4.6 Effect of P, PI, PD and PID Controller
- 4.7 Effect of Adding a zero to a system
- 4.8 Performance Indices(ISE,ITSE,IAE, ITAE)
- 4.9 Approximations of Higher order Systems by Lower order Problems

1.100 The Stability of Linear Control Systems

1.101 The Concept of Stability

1.102 The Routh Hurwitz Stability Criterion

1.103 Relative stability analysis

1.11.0 Root Locus Technique

1.11.1 Angle and Magnitude Criterion

1.11.2 Properties of Root Loci

1.11.3 Step by Step Procedure to Draw Root Locus Diagram

1.11.4 Closed Loop Transfer Fuction and Time Doamin response

1.11.5 Determination of Damping ratio, Gain Margin and Phase Margin from Root Locus

1.11.6 Root Locus for System with transportation Lag.

1.11.7 Sensitivity of Roots of the Characteristic Equation.

1.12.0 Frequency Domain Analysis.

1.12.1 Correlation between Time and frequency response

1.12.2 Frequency Domain Specifications

1.12.3 Polar Plots and inverse Polar plots

1.12.4 Bode Diagrams

1.12.
1.6 Principal factors of Transfer function

1.12.
1.7 Procedure for manual plotting of Bode Diagram

1.12.
1.8 Relative stability Analysis

1.12.4.4 Minimum Phase, Nonminimum phase and All pass systems

1.12.5 Log Magnitude vs Phase plots.

1.12.6 Nyquist Criterion

1.12.
1.11.
1 Mapping Contour and Principle of Argument

1.12.
1.11.
2 Nyquist path and nyquist Plot

1.12.
1.11.
3 Nyquist stability criterion

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1.11.
4 Relative Stability: Gain Margin, and Phase Margin

1.12.7 Closed Loop Frequency Response

1.12.

1.12.

1 Gain Phase Plot

1.12.1.12.1.1 Constant Gain(M)-circles

1.12.1.12.1.2 Constant Phase (N) Circles

1.12.1.12.1.3 Nichols Chart

1.12.8 Sensitivity Analysis in Frequency Domain

Reference Books

1. K. Ogata, “Modern Control Engineering”, 4th Edition, PHI.
2. I. J. Nagrath and M. Gopal, “Control System Engineering”, 4th Edition, New Age.
3. J. J. Distefano, III, A. R. Stubberud and I. J. Williams, “Feedback and Control Systems”, 2nd Edition, TMH, Schaums Outlines.
4. G. F. Franklin, J. D. Powell, A. Emami, Naini, “Feedback Control of Dynamic Systems”, 4th Edition, Pearson Education.
5. B. C. Kuo and F. Golnaraghi, “Automatic Control Systems”, 8th Edition, John Wiley and Sons.
6. S. Ghosh, “Control Systems: Theory and Applications”, 2nd Edition, Pearson.
7. D. RoyChaudhury, “Modern Control Engineering”, 4th Edition, PHI.

Sampled Question Sets

SET-I

CONTROL SYSTEM ENGINEERING-I

Time:-3Hrs **Full Marks:70**

Answer Question no.1 and any five questions from the rest.

Answer all parts of question at one place only

The figures in right hand margin indicate marks.

(Semi log graph papers are allowed)

1 Answer all the following questions briefly (Compulsory) [2x10]

(a) Distinguish between regulator and servo-control problem in control system study. (b)Sketch the underdamped time response of a typical second order feedback control system subjected to a unit step input. State the time domain performance indices.

(c)Prove that a Type-1 system has no steady state error for step input while the steady state error for ramp input decreases for increase of Velocity error Constant(K_v).

(d)Give the equation of intersect of asymptotes in root locus plot.

(e) What is system type number? Explain the practical significance of this number.

(f)Show that the Phase Margin=$\tan^{-1}\frac{2\xi}{\sqrt{\sqrt{1+4\xi^4}-2\xi^2}}$ where ξ is the damping ratio of the standard second order system.

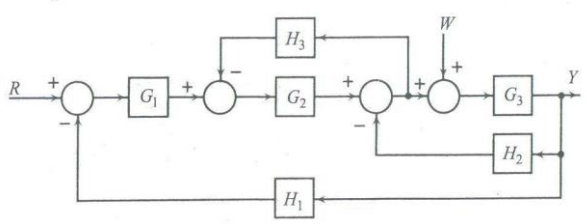
(g)List the advantages and disadvantages of carrying frequency analysis with Nyquist plot.

(h) State the Zeigler-Nichols tuning Rules for PID Controller.

(i) Give all the properties of a minimum phase transfer function.

(j)Explain with sketch the use of drag cup rotor in servo application.

2(a) The Block diagram of a feedback control system is given below. The output Y(s)=C(s)R(s)+D(s)W(s). Find the transfer functions C(s) and D(s). [5]



(b)Describe the construction and working of a two phase motor suitable for use in AC servo systems. [5]

3(a)Show that high loop gain in feedback control system results in

(i)good steady state tracking accuracy (ii)low sensitivity to process parameter variations

(iii)good disturbance signal rejection (iv)good relative stability

What are the factors limiting the gain? [6]

(b)Explain drawing a neat diagram, the principle of operation of a position servo using a synchro system as error transducer. [4]

4.The peak overshoot (%M_p) in a unit feedback control system is specified to be within 20% to 40% range.

(a)Sketch the area in the s-plane in which dominant roots of the systems characteristic equation must lie. This system has a settling time t_s=0.85 sec. [4]

(b)Determine the smallest value of third root such that dominance of the complex roots corresponding to part (a) is preserved. Further, Determine the open loop transfer function of the system if M_p =50% [6]

1.10(a)State the merits and demerits of using static error coefficients. The open loop transfer function in a unity feedback control system, is given by G(s)= ; Find the steady state error of the system using generalized error constants when subjected to an input signal given by $r(t)=1+4t+3t^2$. [5]

(b))In a unity feedback control system, the open loop transfer function is given by

G(s)= ; Using Routh Hurwitz Criterion, determine the range of K for which the given system is stable. [5]

1.11.(a) The Open loop transfer function of a control system is given as
G(s)H(s)= ;

Sketch the Root Locus. Determine the value of K such that damping ratio(ξ) is 0.4. [7]

(b) State the use of Nichol's Chart. [3]

1.12.Using Bode Plot, determine gain crossover frequency, phase crossover frequency, gain margin and phase margin in a unity feedback control system, where, the open loop transfer function is given by

G(s)= . [10]

2. (a)A unity feedback system has open loop transfer function G(s) = ;

Use Nyquist criterion to determine if the system is stable in the closed loop configuration [7]

(b) State the merits and demerits of PI Controller. [3]

CONTROL SYSTEM ENGINEERING(IC323)

Time:-3Hrs Full Marks:70

Answer Question no.1 and any five questions from the rest.

Answer all parts of question at one place only

The figures in right hand margin indicate marks.

(Semi log graph papers are allowed)

1. Answer all the following questions briefly (Compulsory) [2x10]

(a)What are the constraints in developing the transfer function of a device a part of larger system?

(b)The transfer function of a control system is $T(s)=K/[S^2+4S+K]$; Find K if the system is critically damped.

(c) What are the steady state errors of a Type-3 unity feedback system subjected to step input, ramp input and parabolic input?

(d)Explain what do you mean by Root Contours.

(e) The magnitude of frequency response of a second order system is 5 at 0 rad/sec and peaks to $\frac{10}{\sqrt{3}}$ at $5\sqrt{2}$ rad / sec. Determine the transfer function of this underdamped system.

(f)Show that the bandwidth$(\omega_b)=\omega_n[1-2\xi^2+\sqrt{4\xi^4-4\xi^2+2}]$ where ξ is the damping ratio and ω_n is the natural frequency of the standard second order system.

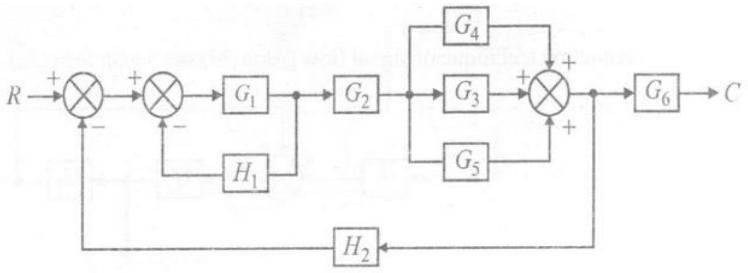
(g) Sketch the constant gain loci for the unity feedback system whose feed forward transfer function is $G(s)=\frac{K}{S(S+1)}$

(h) Show that high loop gain in feedback control system results in good steady-state tracking accuracy

(i) State the use of Nichol's Chart.

(j) State the merits and demerits of PI Controller

2(a) Obtain the signal flow graph representation for a system represented by a block diagram as shown below and determine the overall gain G(s)= – ; [5]



(b)Explain giving a schematic diagram how a synchro pair would be embodied in an AC position control system. [5]

3(a)In a negative feedback control system, calculate separately, the sensitivity of the system transfer function at s= jω=j1.6 rad/sec with respect to

(i)the forward path transfer function G(s) where G(s)= ———

(ii)feedback path transfer function H(s) where H(s)=0.8 [5]

(b)Describe in detail along with a schematic diagram, a typical position control system employing an armature controlled DC Motor with a fixed field separately excited system. Derive the transfer function. [5]

4.(a)What is system type number? Explain the practical significance of this number. [2]

(b) In a unit feedback control system, the open loop transfer function is given by G(s)= ———

By what factor should the amplifier gain k be multiplied so that the damping ratio(ξ) is enhanced from 0.35 to 0.91.10 [8]

5(a)In a unit feedback control system, the open loop transfer function is given by

G(s)= ———————; Find the static error coefficients (K_p, K_v, and Ka) and the steady state error of the system when subjected to an input signal given by $r(t)=10+20t+30t^2$. [5]

(b))In a unit feedback control system, the open loop transfer function is given by

G(s)= ———————; Using Routh Hurwitz Criterion, determine the range of K for which the given system is stable. [5]

1.11. (a)The Open loop transfer function of a control system is given as G(s)H(s)= ———;

Sketch the Root Locus. [6]

(b)Given the open loop frequency response G(jω) = U+ jV;

Obtain the radii and center locations of constant M and N circles [4]

7(a)Define minimum phase, non-minimum phase and All pass system. [2]

(b)Draw the Bode Plot of the open loop transfer function of a feedback system given by

G(s)H(s)= ; Also determine the system Stability. [8]

8(a)A unity feedback system has open loop transfer function G(s) = ;

Use Nyquist criterion to determine if the system is stable in the closed loop configuration. [7]

(b)Describe two tuning methods, one based on ultimate gain and the other based on process reaction curve. [3]

SET-III

Time:-3Hrs Full Marks:70

Answer Question no.1 and any five questions from the rest.

The figures in right hand margin indicate marks.

(Semi log graph papers are allowed)

1.Answer all the following questions briefly (Compulsory) [2x10]

(a)Sketch the underdamped time response of a typical second order feedback control system subjected to a unit step input. State the time domain performance indices.

(b)The transfer function of a control system is $T(s)=K/[S^2+4S+K]$; Find K if the system is critically damped.

(c)Prove that a Type-1 system has no steady state error for step input while the steady state error for ramp input decreases for increase of Velocity Error Constant (K_v).

(d)Give the equation of intersect of asymptotes in root locus plot.

(e)Explain what do you mean by Root Contours.

(f)Show that the Phase Margin=$\tan^{-1}$ where ξ is the damping ratio of the standard second order system.

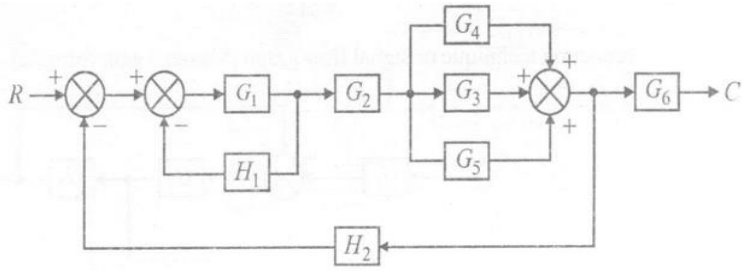
(g)List the advantages and disadvantages of carrying frequency analysis with Nyquist plot.

(h) State the Zeigler-Nichols tuning Rules for PID Controller.

(i) Give all the properties of a minimum phase transfer function.

(j)Explain with sketch the use of drag cup rotor in servo application.

2(a)Obtain the signal flow graph representation for a system represented by a block diagram as shown below and determine the overall gain G(s)= – ; [6]



(b)Describe the construction and working of a two phase motor suitable for use in AC servo systems. [4]

3(a)Show that high loop gain in feedback control system results in

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(iii)good disturbance signal rejection (iv)good relative stability

What are the factors limiting the gain? [5]

(b)Explain drawing a neat diagram, the principle of operation of a position servo using a synchro system as error transducer. [5]

4.The peak overshoot (%M_p) in a unit feedback control system is specified to be within 20% to 40% range.

(a)Sketch the area in the s-plane in which dominant roots of the systems characteristic equation must lie. This system has a settling time t_s=0.85 sec. [4]

(b)Determine the smallest value of 3rd root such that dominance of the complex roots corresponding to part (a) is preserved. Further, Determine the open loop transfer function of the system if M_p=50% [6]

1.10(a)State the merits and demerits of using static error coefficients. The open loop transfer function in a unity feedback control system, is given by G(s)= ; Find the steady state error of the system using generalized error constants when subjected to an input signal given by $r(t)=1+4t+3t^2$. [5]

(b)In a unity feedback control system, the open loop transfer function is given by G(s)= ; Using Routh Hurwitz Criterion, determine the range of K for which the given system is stable. [5]

1.11.(a) The Open loop transfer function of a control system is given as G(s)H(s)= ;

Sketch the Root Locus. Determine the value of K such that damping ratio(ξ) is 0.4. [8]

(b) State the use of Nichol's Chart. [2]

1.12.Using Bode Plot, determine gain crossover frequency, phase crossover frequency, gain margin and phase margin in a unity feedback control system, where, the open loop transfer function is given by G(s)= . [10]

2. A unity feedback system has open loop transfer function G(s) = ; Use Nyquist criterion to determine if the system is stable in the closed loop configuration. [10]

Set-IV

Sub: **Control System Engineering (3:1:0)**

Time Duration : Two Hours Date 2.10.2013 Maximum 20 Marks

A.The figures in the right hand margin indicate marks Answer any four including Question No.1 B.The symbols carry usual meaning

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1. Answer the following questions (Compulsory) [5x1]

(a) If $\frac{Y(s)}{U(s)} \quad \frac{s\ 3}{s^2\ \ 3s\ 2}$ obtain the SFG representation of this transfer function.

(b) Experimental measurements yield a plot of the magnitude of the frequency response function with a resonance peak 1.35 at a frequency of 10rad/sec.

(a)Estimate and $_n$ of the dominating system poles

(c)Draw the schematic diagram of a DC closed loop position control system consisting of

(I) a pair of Potentiometers (II)Amplifier (III)Armature controlled DC Servomotor

(IV)Gear Train as major component and explain the operation of this system

(d) .The open loop transfer function of a unity feedback control system is given by $G(s)=K/[(s+2)(s+4)(s^2+6s+25)]$; By applying Routh-Hurwitz criterion determine the range of K for which the closed loop system will be stable:

(e)Draw the polar plot for the transfer function of the system $G(s)H(s)=10/[s(s+1)^2]$.

2(a)Consider a -ve unity feedback system with following OLTF. Obtain peak overshoot, damped frequency of oscillation, settling time on 2% tolerance band and response of the system to unit step input.

( ) = (.)

(b)The OLTF of a system is G(s)H(s)=100/[s(s+100)]. (i)Obtain Static and Dynamic error Constants. (ii)If the input is $r(t)=A+Bt+Ct^2$, obtain the steady state error and the dynamic error. [3+2]

3.Sketch the complete Bode plot of the unity feedback system whose open loop frequency function

$\frac{10}{s(0.1s+1)(0.05s+1)}$; Determine the GM, PM and open loop gain for a GM of 20db. [5]

4.Given the G(s)= $\frac{K}{s(s+1)}$, and $H(s)=(s+4)^{-1}$. Sketch the root locus of the system.

(i)Determine the value of K for which the system is at the verge of instability

(ii)For the damping ratio 0.34, determine the value of K and the GM. [5]

1.10(a) Show that the bandwidth of a linear standard second order control system = ω_n $[1-2\xi+4\xi-4\xi+2]$ where ξ is the damping ratio and ω_n is the natural frequency of system. What will be the resonant peak for the system whose transfer function is $5/(s^2+2s+5)$

(b)Determine the critical value of K for stability of a unity feedback system with loop transfer function(S)=K/(S-1) using Nyquist stability criterion. [3+2]